

THE IMPEDANCE-FREQUENCY CHARACTERISTICS OF QUARTZ CRYSTALS

A THESIS

Presented to  
the Faculty of the Division of Graduate Studies  
Georgia Institute of Technology

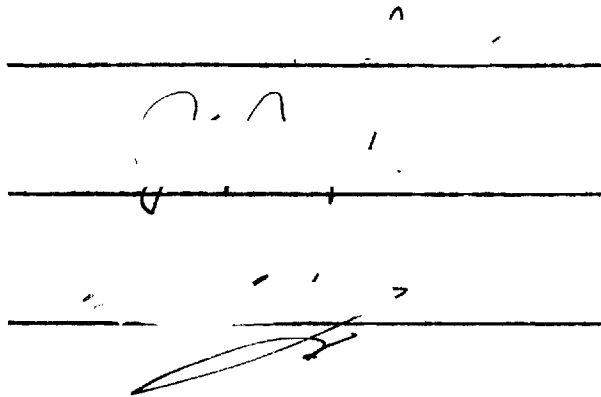
In Partial Fulfillment  
of the Requirements for the Degree  
Master of Science in Physics

by  
Frederick Dixon

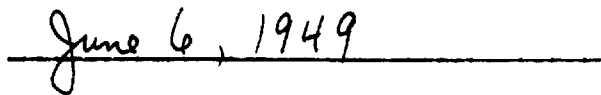
June 1949

## THE IMPEDANCE-FREQUENCY CHARACTERISTICS OF QUARTZ CRYSTALS

Approved:

A handwritten signature, possibly "J. N.", is written across three horizontal lines. The signature is written in a cursive style with a large loop at the end.

Date Approved by Chairman

The date "June 6, 1949" is handwritten in cursive across a horizontal line.

## ACKNOWLEDGMENTS

This thesis is justly a product of interest engendered during the year 1947 by association with Dr. William A. Edson, Mr. Vernon R. Widerquist, and Miss Catherine Yoe, in connection with Project No. 106-6 at the State Engineering Experiment Station, Georgia Institute of Technology. Professor Edson's continued activity in research on crystal circuit problems has proved a useful general stimulant for completing the paper, and his suggestions on many specific points of theory and presentation have been of considerable help. In submitting this work I wish to express especial thanks, also, to Mr. Walter J. Cleveland for invaluable assistance in preparing and reproducing the figures.

## PREFACE

The literature on the electrical characteristics of piezoelectric crystals was much enriched in 1946 with the appearance of Cady's Piezoelectricity and Heising's Quartz Crystals for Electrical Circuits. The latter, dealing specifically with the most widely used type of crystal, is a "must" for the engineer engaged in the design of practical crystal-controlled oscillators.

However, the scope of both texts is rather broad, with the result that the treatments of the equivalent electrical network of a crystal suffer from being insufficiently complete (or obvious) for the average college student. In Cady the analysis proceeds largely by graphical means, while in Heising the immediate concern is with crystals connected in standard oscillator circuits. Thus, there still remains room for a really detailed examination, by complex algebra, of the equivalent electrical impedance existing between the terminals of a crystal considered as an independent unit.

A few remarks on the general approach are in order. The crystal has been considered purely from the standpoint of its electrical representation. Naturally, the results derived will be valid only insofar as the electrical representation is itself valid. Practically, however, it would appear that present-day methods of measuring the equivalent parameters of a crystal are subject to errors very nearly as great as those which might be introduced by such sources as the possible functional dependence of the crystal "constants" upon frequency or amplitude of vibration.

Quartz has been emphasized in that, wherever necessary, the known limiting values of quartz crystal constants are called upon to establish validity for mathematical approximations. This practice is justified in view of the almost exclusive use of quartz for oscillator stabilization and control and in other devices requiring high-Q resonant elements. Again, however, insofar as other types of crystals satisfy the same conditions for the approximations, their electrical behavior may be regarded as quantitatively defined by the equations and curves presented.

Because of the absence of "practical" material, the work appears to be a tabulation of formulas. This, however, is an intentional result; no attempt has been made to consider specific problems. Nevertheless, many important applications of the mathematics established herein suggest themselves and may be taken up as separate studies. Of particular interest is the virtually unexplored behavior of crystals when shunted by excessively large padding capacitance.

## TABLE OF CONTENTS

	PAGE
Acknowledgments .....	iii
Preface .....	iv

## I. ELECTRICAL DEFINITIONS OF A CRYSTAL

1-1. The Equivalent Network of a Crystal .....	1
1-2. Primary Definitions .....	3
1-3. The Practical Equivalent Network of a Crystal .....	5
1-4. Secondary Definitions .....	6
1-5. The Complex-Algebra Equivalent of a Crystal .....	8
1-6. The Input Impedance of a Crystal .....	9
1-7. The Input Admittance of a Crystal .....	10

## II. THE EQUIVALENT SERIES IMPEDANCE OF A CRYSTAL

2-1. Basic Form of the Resistance and Reactance Functions .....	12
2-2. The Equation of Phase Angle and Frequency.....	13
2-3. Development of the Resistance Function .....	13
2-4. Relationship between Frequency and Phase Angle .....	14
2-5. Relationship between Maximum Phase Angle and Normalized Capacitance Ratio .....	15
2-6. Simplification of the Frequency-Phase Angle Relationship .....	17
2-7. Parametric Forms of the Impedance Functions of a Crystal .....	17
2-8. The Direct Impedance Functions of a Crystal .....	19

## III. RAPID PLOTTING OF THE CRYSTAL IMPEDANCE FUNCTIONS

3-1. Resistance Maximum .....	25
3-2. Series and Parallel Resonance .....	28
3-3. Double Resistance Root .....	30
3-4. Reactance Maximum and Minimum .....	31
3-5. Tabulation of Critical-Point Functions .....	33
3-6. Method of Plotting Crystal Impedance Characteristics .....	45

## IV. THE EQUIVALENT ADMITTANCE OF A CRYSTAL

4-1. The Admittance Functions of a Crystal .....	49
4-2. Crystal Conductance and Susceptance .....	50
4-3. Polar Admittance Curve for the RLC Branch .....	51
4-4. Polar Curves of Total Admittance .....	53
4-5. Resonance Coalescence .....	56

## TABLE OF CONTENTS

(continued)

	PAGE
V. DERIVATIVE FUNCTIONS OF A CRYSTAL	
5-1. Derivatives with Respect to Frequency .....	60
5-2. Derivatives with Respect to Phase Angle .....	61
5-3. Derivatives with Respect to External Shunt Capacitance .....	62
5-4. Rate of Change of Phase Angle with Frequency .....	62
5-5. Rate of Change of Frequency with Shunt Capacitance .....	63
5-6. Rate of Change of Resistance with Frequency .....	66
5-7. Rate of Change of Reactance with Frequency .....	66
5-8. Rate of Change of Impedance Magnitude with Frequency .....	67
5-9. Rate of Change of Phase Angle with Impedance Level .....	68
VI. APPLICATIONS	
6-1. General Effects of Shunt Capacitance on Crystal Performance...	70
6-2. Solution of Circuit Problems .....	71
6-3. Methodology .....	72
Bibliography .....	73
APPENDIX A. CRYSTAL RESONANCES .....	74
APPENDIX B. THE "Q" OF A CRYSTAL .....	77

## THE IMPEDANCE-FREQUENCY CHARACTERISTICS OF QUARTZ CRYSTALS

### I. ELECTRICAL DEFINITIONS OF A CRYSTAL

Piezoelectric crystals are today in wide use as resonators, both in filters and in oscillator circuits. Strictly speaking, a piezoelectric resonator is an electromechanical transducer, and its behavior can be thoroughly analyzed only in terms of the direct and converse piezoelectric effects, the understanding of which involves knowledge of molecular structures. However, the characteristics of piezoelectric crystals which are important in electrical engineering applications may, on the whole, be determined satisfactorily from purely electrical considerations. This point of view is followed exclusively in the present treatment of quartz crystal characteristics.

#### 1-1. The Equivalent Network of a Crystal

The network shown in Figure 1 is the familiar equivalent electrical representation of a crystal possessing piezoelectric properties. The series-arm parameters  $\underline{L}_S$ ,  $\underline{C}_S$ , and  $\underline{R}_S$  correspond to the mechanical inertia, compliance, and frictional resistance, respectively, of the vibrating crystalline body.  $\underline{C}_0$  is a true electrical capacitance, being that of the holder plates, with the crystal material as dielectric.

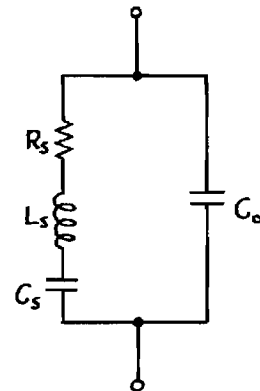


Figure 1.  
The Equivalent Electrical  
Circuit of a Crystal.



Whenever appreciable dielectric losses are present, an additional resistance must be shown in series with  $C_0$ . However, in practice this is seldom necessary.

Figure 2 illustrates the typically complex behavior of crystal impedance over a broad frequency spectrum. It must be understood that the representation of Figure 1 applies when the crystal is vibrating at a frequency close to only one of the resonances. A set of distinct values for the equivalent series-arm

parameters holds in the neighborhood of each vibrational mode or overtone resonance. The additional modes of vibration may be represented by RLC branches connected in parallel to the circuit of Figure 1—one such branch for each resonance indicated

in Figure 2. However, except where resonances fall extremely close to-

gether, the impedances of all branches but the one  $R_S L_S C_S$  of interest will be high, and can generally be neglected.

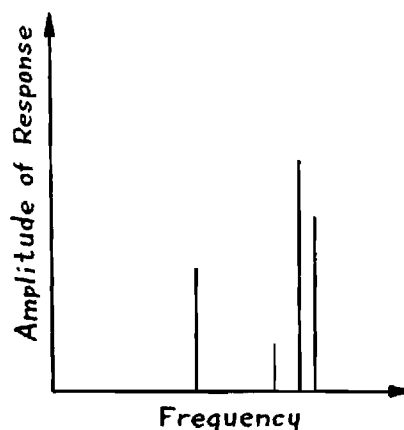


Figure 2.  
Typical Frequency Spectrum  
of a Quartz Plate.

Manufactured plug-in crystals have an air gap between one of the electrical faces of the crystal and its electrode. Such a series air-capacitance may be included in the simpler representation of Figure 1, with no loss of validity, merely by adjusting the values of all the equivalent parameters.<sup>1</sup> The air gap is, therefore, omitted from considera-

---

<sup>1</sup>A thorough treatment of the gap effects is given in Chapters XIII and XIV of W. G. Cady, Piezoelectricity (New York: McGraw-Hill Book Company, 1946).

tion in the ensuing treatment of the crystal network.

The equivalent parameters will be regarded as constants, independent of frequency (over the permissible frequency range) and independent of the potential difference between the holder plates or of the amplitude of vibration of the crystal. It is well to point out that whereas this assumption appears perfectly justified for quartz and tourmaline, Rochelle salt exhibits nonlinearity between mechanical stress and strain, with corresponding electrical nonlinearity for which allowance must be made. As the present paper deals specifically with the properties of quartz crystals, the above consideration does not arise. However, it introduces a limitation in attempting to extend the results obtained to other types of crystals.

## 1-2. Primary Definitions

Three formal definitions are made now to facilitate the mathematical analysis. Each involves merely a grouping of two or more of the four equivalent parameters of Figure 1, and at present no particular significance will be attached to any of the new quantities.

a. Capacitance Ratio. The ratio of the shunt to the series capacitance of a crystal is shown by Cady<sup>2</sup> to be equivalent to the ratio of Electrical to Mechanical Energy during vibration. He refers to crystals having intrinsically small values of this ratio as "high activity crystals." For our purposes the capacitance ratio is simply a convenient figure to specify in ordering crystals for a particular application. We define:

---

<sup>2</sup>Ibid., p. 353.

$$n_o = C_o/C_s \quad (1.1)$$

For quartz crystals this capacitance ratio is invariably greater than 100, the value 125 being the conventional lower limit for  $n_o$ .<sup>3</sup>

b. Reference Frequency. The resonant frequency of the series arm of Figure 1 will be employed as a reference frequency in the subsequent mathematical work. This frequency is:<sup>4</sup>

$$\omega_s = \frac{1}{\sqrt{L_s C_s}} \quad (1.2)$$

c. Quality Factor. The usual engineering approximation for the selectivity or quality factor,  $Q$ , of a series resonant branch containing resistance is used in connection with the series arm parameters of Figure 1. Thus,

$$Q_s = \frac{\omega_s L_s}{R_s} = \frac{1}{\omega_s C_s R_s} = \sqrt{\frac{L_s}{C_s R_s^2}} \quad (1.3)$$

Because crystals are characterized by values of  $Q$  in the order of  $10^5$ , it is not surprising that any approximation to the true  $Q$ , such as (1.3), turns out to be quite exact.<sup>5</sup> It will be necessary to assume that the quantity defined by (1.3) is large. However, in all other respects,  $Q_s$

---

<sup>3</sup>Ibid., p. 349.

<sup>4</sup>The term "frequency" usually refers to the number of cycles of a periodic variation which occur in one second. More generally, frequency is the number of times any given phenomenon repeats itself in unit time. Under this definition, the symbol  $f$  denotes cyclic frequency, the symbol  $\omega$  radian frequency; the less specific term frequency applies properly to both quantities.

<sup>5</sup>See Appendix B for a discussion of Quality Factors.

will be treated purely as a number computed from the equivalent parameters of the crystal and chosen only for mathematical convenience.

### 1-3. The Practical Equivalent Network of a Crystal

Whenever a crystal is connected to another electrical device, in effect some capacitance is placed in parallel with the crystal terminals. In many cases it is desirable, and in no case is it detrimental to analysis, to regard this additional shunting capacitance as part of the crystal proper. Figure 3 shows the external capacitance  $C_x$  combined with the normal shunt capacitance of the crystal to form a total parallel capacitance of

$$C_y = C_o + C_x \quad (1.4)$$

The addition of external capacitance is frequently made deliberately, for one reason or another. It is clear that this type of pad-

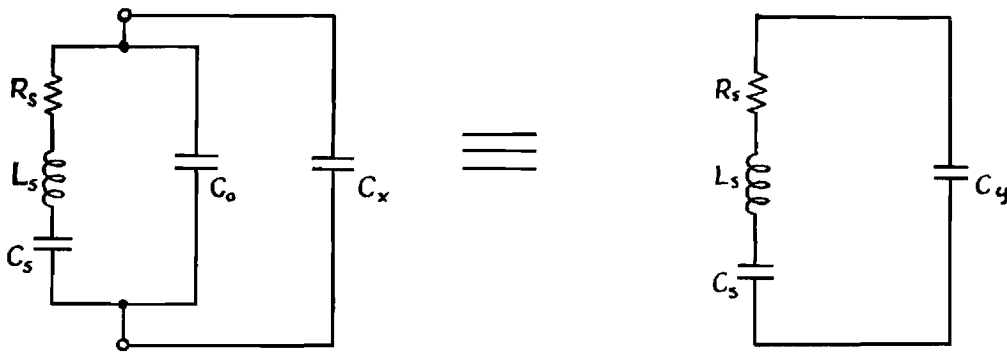


Figure 3. The Practical Equivalent Circuit of a Crystal.  
External Shunting Capacitance Is Considered  
as Part of the Resonant Element Combination.

ding can be considered independently of the remainder of the circuit to which the crystal is connected, so that it will be useful to ascertain

how the properties of a crystal vary with respect to the amount of shunt capacitance present.

#### 1-4. Secondary Definitions

a. Practical Capacitance Ratio. Considering the total shunt capacitance  $C_y$  as belonging to the crystal, the practical capacitance ratio is written:

$$n = C_y / C_s \quad (1.5)$$

To emphasize the distinction with (1.1),  $n_o$  will be referred to as the "intrinsic" capacitance ratio of a crystal. It is evident that, whereas  $n_o$  might have an upper limiting value for a given kind of substance, the practical ratio  $n$  has no upper limit except that dictated by engineering practice. However,  $n$  may never be less than  $n_o$ .

b. Normalized Capacitance Ratio. A composite parameter which will appear frequently in the crystal circuit analysis is the capacitance ratio divided by some form of the crystal  $Q$ . We shall call this parameter,  $N$ , the "normalized capacitance ratio," defined as:

$$N = n / Q_s \quad (1.6)$$

When only the crystal proper is considered, (1.6) gives the "normalized intrinsic capacitance ratio,"  $N_o = n_o / Q_s$ . The reciprocal of this intrinsic parameter  $N_o$  was introduced by Fair<sup>6</sup> as a suitable modified "figure of merit" for crystals, denoted by the symbol  $M$ .

---

<sup>6</sup>R. A. Heising, Quartz Crystals for Electrical Circuits (New York: D. Van Nostrand Company, 1946), p. 395.

c. Antiresonance Factor. As an aid in transformations, the following auxiliary quantity is defined:

$$m = \sqrt{1 + \frac{1}{n}} \quad (1.7)$$

m may be referred to as the "antiresonance factor," since the antiresonant frequency of a crystal is shown to be very closely equal to  $\underline{m}\underline{w}_s$ .

d. Frequency Factor. In the ensuing mathematical derivations, all frequencies will be referred to the value  $\underline{w}_s$  by means of the factor c.

Thus,

$$c = w/w_s \quad (1.8)$$

Note that c is a dimensionless quantity.

e. Fractional Frequency Deviation. Because the analysis is restricted to frequencies in the neighborhood of a particular resonance, w will not differ greatly from  $\underline{w}_s$  and c will be approximately unity in value. It follows that the frequency factor may be written:

$$c = 1 + d \quad (1.9)$$

in which d is invariably a quantity small compared to one. The term d will be referred to as the "fractional frequency deviation," since it is the fraction of  $\underline{w}_s$  by which w differs from  $\underline{w}_s$ . The deviation d will normally be expressed in parts per million and may often be regarded as an infinitesimal.

f. Normalized Frequency Deviation. A somewhat arbitrary unit, but one which proves extremely useful in analysis, is obtained from multiplying the fractional frequency deviation by the approximate crystal Q

defined in (1.2). Thus,

$$D = Q_S d \quad (1.10)$$

Through the use of this "normalized frequency deviation"  $\underline{D}$ , rather than the straight fractional deviation  $\underline{d}$ , many of the engineering formulas derived later can be plotted independently of the crystal  $Q$ . The term  $\underline{D}$  will quite typically range between values of zero and ten.

#### 1-5. The Complex-Algebra Equivalent of a Crystal

Figure 4 shows the practical equivalent network of a crystal arranged in a form suitable for analysis by complex algebra. Using

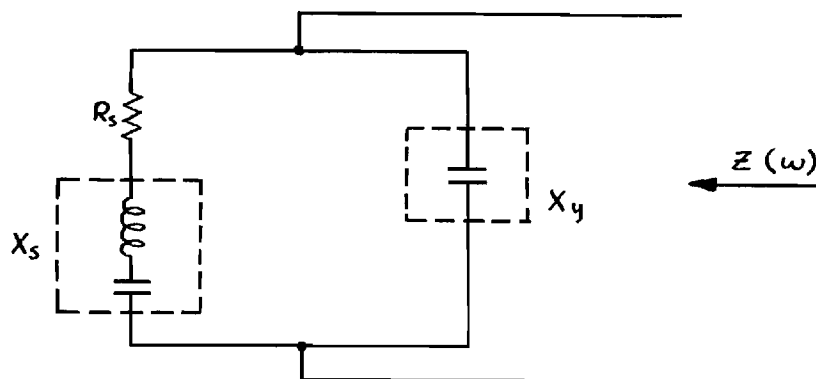


Figure 4. The Complex-Algebra Equivalent of a Crystal.

the definitions of Sections 1-2 and 1-4, the following transformations can be made:

$$X_y = \frac{-1}{wC_y} = \frac{-1}{w_s C_s c n} = \frac{-R_s Q_s}{c n} = -\frac{R_s}{c n} \quad (1.11)$$

$$X_s = wL_s - \frac{1}{wC_s} = \frac{R_s Q_s (c^2 - 1)}{c} \quad (1.12)$$

$$(X_s + X_y) = \frac{R_s Q_s (c^2 - m^2)}{c} \quad (1.13)$$

### 1-6. The Input Impedance of a Crystal

The impedance seen looking into the terminals of the crystal shown in Figure 4 is evidently:

$$Z(w) = R(w) + jX(w) = \frac{jX_y(R_s + jX_s)}{R_s + j(X_s + X_y)} \quad (1.14)$$

Rationalizing, and separating reals and imaginaries,

$$R(w) = \frac{R_s X_y^2}{R_s^2 + (X_s + X_y)^2} \quad (1.15)$$

$$X(w) = \frac{X_y [R_s^2 + X_s(X_s + X_y)]}{R_s^2 + (X_s + X_y)^2} \quad (1.16)$$

Now let

$$u = \frac{X(w)}{R(w)} = \frac{R_s^2 + X_s(X_s + X_y)}{R_s X_y} \quad (1.17)$$

Then the impedance function may be written:

$$Z(w) = |Z(w)| \angle \underline{\theta} \quad (1.18)$$

where

$$|Z(w)| = \sqrt{R(w)^2 + X(w)^2} \quad (1.19)$$

and

$$\theta = \arctan u \quad (1.20)$$

The input impedance of a crystal is usually called its "equivalent series impedance." It is clear that this impedance is defined if  $R(w)$  and  $X(w)$ , or  $|Z(w)|$  and  $\underline{\theta}$ , are defined. In the subsequent development, the four quantities mentioned above will be referred to collectively as the "impedance functions of a crystal."



### 1-7. The Input Admittance of a Crystal

Admittance being defined as the reciprocal of impedance, it follows from (1.14) that

$$Y(w) = G(w) + jB(w) = \frac{R_s + j(X_s + X_y)}{jX_y(R_s + jX_s)} \quad (1.21)$$

Rationalizing, and separating reals and imaginaries,

$$G(w) = \frac{R_s}{R_s^2 + X_s^2} \quad (1.22)$$

$$B(w) = -\left[ \frac{X_s}{R_s^2 + X_s^2} + \frac{1}{X_y} \right] \quad (1.23)$$

Admittance functions are, for some reason, not as familiar to the average engineer as impedance functions. However, where branches in parallel are involved, the equivalent admittance of a network often proves easier to analyze, from a mathematical standpoint, than the corresponding equivalent series impedance.

Just as the total impedance of a group of elements in series is the vector sum of the individual impedances (resistance or reactance); so the total admittance of elements in parallel is the vector sum of the individual admittances (conductance or susceptance). The total conductance of a crystal should, therefore, be simply the conductance of the series  $R_s L_s C_s$  branch; while the total susceptance should be the sum of the susceptances of  $R_s L_s C_s$  and of  $C_y$ . Equation (1.22) will be recognized as the conductance of a series branch containing resistance  $R_s$  and reactance  $X_s$ . Similarly, the first term on the right of (1.23) is the susceptance of such a series branch; while, the second term is obviously

the susceptance of a branch containing only a reactance  $\underline{X}_y$ .

Although the impedance viewpoint is given greater emphasis in this paper, reference will be made to the admittance viewpoint wherever it offers marked advantages.

## II. THE EQUIVALENT SERIES IMPEDANCE OF A CRYSTAL

In this chapter the impedance functions of a crystal are developed in terms of the crystal constants of Figure 3 and in terms of a variable involving frequency. The results are brought into forms suitable for analysis of various types of circuits which include the crystal as a part.

### 2-1. Basic Form of the Resistance and Reactance Functions

Substituting (1.11) and (1.13) into (1.15), the resistance function is written:

$$R(w) = \frac{R_s}{(D)_R} \quad (2.1)$$

where

$$(D)_R = N^2 \left[ c^2 + Q_s^2 (m^2 - c^2)^2 \right] \quad (2.2)$$

Similarly, using (1.11), (1.12), and (1.13), the reactance function (1.16) becomes:

$$X(w) = R_s \frac{(N)_X}{(D)_X} \quad (2.3)$$

where

$$(N)_X = Q_s^2 (c^2 - 1)(m^2 - c^2) - c^2 \quad (2.4)$$

$$(D)_X = cN \left[ c^2 + Q_s^2 (m^2 - c^2)^2 \right] \quad (2.5)$$

It should be noted that the functions  $R(w)$  and  $X(w)$  are now expressed wholly in terms of dimensionless numbers, except for the intrinsic crystal resistance  $\underline{R}_s$ , which may be looked upon as a sort of basic impedance level.

## 2-2. The Equation of Phase Angle and Frequency

Employing once more the transformations of Section 1-5, Equation (1.17) is written:

$$u = \frac{N}{c} \left[ Q_s^2 (c^2 - 1)(m^2 - c^2) - c^2 \right] \quad (2.6)$$

Upon expansion in powers of  $\underline{c}$ , this becomes:

$$c^4 - c^2(m^2 + 1 - \frac{1}{Q_s^2}) + \frac{cu}{NQ_s^2} + m^2 = 0 \quad (2.7)$$

Equation (2.7) is an implicit form of the fundamental relationship between the frequency factor,  $\underline{c}$ , and the tangent,  $\underline{u}$ , of the impedance phase angle.

## 2-3. Development of the Resistance Function

Upon expanding (2.2), there results:

$$(D)_R = N^2 Q_s^2 \left( \frac{c^2}{Q_s^2} + m^4 - 2m^2 c^2 + c^4 \right) \quad (2.8)$$

Substituting for  $\underline{c}^4$  from (2.7),

$$\begin{aligned} (D)_R &= N^2 Q_s^2 \left[ m^2(m^2 - 1) - \frac{cu}{NQ_s^2} - c^2(m^2 - 1) \right] \\ &= (NQ_s + 1) - Nuc - NQ_s c^2 \end{aligned} \quad (2.9)$$

since by (1.7) and (1.6) the quantity  $(m^2 - 1) = \frac{1}{n} = \frac{1}{NQ_s}$ .

Now introducing the fractional frequency deviation term from (1.9), Equation (2.9) becomes:

$$\begin{aligned} (D)_R &= 1 - Nu(1 + d) - NQ_s d(2 + d) \\ &= 1 - Nu(1 + d) - ND(2 + d) \end{aligned} \quad (2.10)$$

using the normalized frequency deviation of (1.10).

If the frequency term  $\underline{d}$  is sufficiently small, (2.10) may be reduced to the approximate form,

$$(D)_R = 1 - Nu - 2ND \quad (2.11)$$

The resistance function (2.1) will then be specified in terms of the crystal constant  $\underline{N}$  and a single other independent variable as soon as  $\underline{u}$  is expressed as a function of  $\underline{D}$ , or vice versa.

#### 2-4. Relationship between Frequency and Phase Angle

The solution of the Frequency-Phase-Angle Equation (2.7) is facilitated by making the approximation,<sup>7</sup>

$$c^4 = 1 + 4d + 6d^2 \quad (2.12)$$

which is valid for  $\underline{d}$  sufficiently small. Then (2.7) becomes, through further use of (1.9):

$$\begin{aligned} (1 + 4d + 6d^2) - (1 + 2d + d^2)(m^2 + 1 - \frac{1}{Q_s^2}) \\ + \frac{(1 + d)u}{NQ_s^2} + m^2 = 0 \end{aligned} \quad (2.13)$$

Expanding and collecting terms,

$$0 = \frac{d^2}{2}(4NQ_s - 1 + \frac{N}{Q_s}) - d(1 - \frac{N}{Q_s} - \frac{u}{2Q_s}) + \frac{(N + u)}{2Q_s} \quad (2.14)$$

Solving this quadratic in  $\underline{d}$ , there results:

---

<sup>7</sup>This consists of discarding the terms  $4d^3$  and  $d^4$  in the binomial expansion of  $(1 + d)^4$ .

$$d = \frac{(1 - \frac{N}{Q_s} - \frac{u}{2Q_s}) \pm \sqrt{\Delta(d)}}{4NQ_s - 1 - \frac{N}{Q_s}} \quad (2.15)$$

where

$$\Delta(d) = \left[1 - \frac{(N+u)}{2Q_s}\right]^2 - \frac{(N+u)}{Q_s}(4NQ_s - 1 + \frac{N}{Q_s}) \quad (2.16)$$

## 2-5. Relationship between Maximum Phase Angle and Normalized Capacitance Ratio

Expanding the discriminant (2.16),

$$\Delta(d) = -N^2(4 + \frac{3}{4Q_s^2}) - Nu(4 - \frac{1}{2Q_s^2}) + (1 + \frac{u^2}{4Q_s^2}) \quad (2.17)$$

Considerable simplification of this result is made possible by virtue of the notably high values of  $Q$  possessed by quartz crystals. Neglecting the constant terms in  $\frac{1}{Q_s^2}$ , (2.17) becomes:

$$\Delta(d) = -4N^2 - 4Nu + (1 + \frac{u^2}{4Q_s^2}) \quad (2.18)$$

The term in  $u^2$  is retained since the tangent function of an angle is not bounded.

If only real frequencies are considered, it is necessary that the discriminant of (2.15) be either positive or zero. The limitation is therefore imposed that  $N$  be related to the other quantities in (2.18) according to the inequality:

$$N \leq -\frac{u}{2} \pm \frac{u}{2} \sqrt{1 + \frac{1}{u^2} + \frac{1}{4Q_s^2}} \quad (2.19)$$

---

<sup>8</sup>Equation (2.19) is obtained by solving the inequality  $\Delta(d) \geq 0$  as though it were a quadratic equality in  $N$ , and then applying the condition.

The term in  $Q_s^2$  under the radical may now be neglected; and since  $\underline{N}$  is necessarily a positive number only the positive square root need be considered. Thus, from (2.19),

$$N_{\max} = \frac{1}{2} \left[ \sqrt{1 + u^2} - u \right] \quad (2.20)$$

Equation (2.20) shows that for every value of crystal phase angle there is an upper limit to the values which the normalized capacitance ratio  $\underline{N}$  may have and still permit the crystal to operate at the desired phase angle. It is also evident, from the manner in which (2.19) was derived, that when  $\underline{N}$  takes on the particular value given by (2.20) the discriminant (2.18) is zero and (2.15) has only one root. This fact is, at the moment, principally of mathematical interest; however, its physical significance is made apparent in Section 3-3 of Chapter III.

Equation (2.20) may be solved for  $\underline{u}$ , yielding:

$$u_{\max} = \frac{1 - 4N^2}{4N} \quad (2.21a)$$

An alternative useful form is obtained by denormalizing the denominator:

$$\frac{u_{\max}}{Q_s} = \frac{1 - 4N^2}{4n} \quad (2.21b)$$

The first relationship shows that for a given value of the crystal parameter  $\underline{N}$ , the impedance phase angle can never become greater than the number computed by taking the inverse tangent of (2.21). With the (for quartz) very extreme values  $n = 125$  and  $Q_s = 10^6$ , the normalized capacitance ratio will be  $1.25 \times 10^{-4}$ , giving a maximum phase angle tangent of 2000. This number may be considered the ultimate limit of  $\underline{u}$  for quartz crystals.

## 2-6. Simplification of the Frequency-Phase Angle Relationship

The remaining term in (2.18) which contains  $\underline{Q}_s$  may now be discarded since (2.21b) shows that  $\frac{u}{\underline{Q}_s}$  cannot ever be greater than the relatively small number  $\frac{1}{\underline{L}n}$ . Hence,

$$\begin{aligned}\Delta(d) &= 1 - \underline{L}N^2 - \underline{L}Nu \\ &= 1 - \underline{L}N(N + u)\end{aligned}\tag{2.22}$$

Similarly, for  $\underline{Q}_s \gg 1$ , (2.15) becomes:

$$d = \frac{1 \pm \sqrt{1 - \underline{L}N(N + u)}}{\underline{L}N\underline{Q}_s}\tag{2.23}$$

or, using the normalized frequency term, defined in Section 1-4,

$$D = \frac{1 \pm \sqrt{1 - \underline{L}N(N + u)}}{\underline{L}N}\tag{2.24}$$

Equation (2.24) may be solved for  $u$  to give:

$$u = -(\underline{L}ND^2 - 2D + N) = 2D(1 - \underline{L}ND) - N\tag{2.25}$$

Thus, the relationship between the normalized frequency term and the tangent of the impedance phase angle is seen to depend, for all practical purposes, only on the one crystal parameter  $\underline{N}$ .

## 2-7. Parametric Forms of the Impedance Functions of a Crystal

The phase angle at which a crystal operates proves to be an extremely useful quantity in connection with several circuit problems. From a mathematical standpoint, the tangent of this phase angle is just as useful, and is particularly convenient for manipulation in the crystal functions. It is possible to express the dependence of the four im-



pedance functions on frequency through the parametric variable  $\underline{u}$ , as follows.

Substituting the expression (2.24) for  $\underline{D}$  in (2.11), and inserting the result into (2.1),

$$R(w) = \frac{2R_s}{1 - 2Nu \mp \sqrt{1 - 4N(N + u)}} \quad (2.26)$$

From (1.17),

$$X(w) = u \cdot R(w) \quad (2.27)$$

while, from (1.19),

$$|Z(w)| = \sqrt{1 + u^2} \cdot R(w) \quad (2.28)$$

Finally, (1.20) is repeated here for convenience:

$$\theta = \arctan u \quad (2.29)$$

The relationship between frequency and phase angle is, perhaps, better shown by rewriting (2.25) in the form:

$$\left[ D - \frac{1}{4N} \right]^2 = - \frac{1}{4N} \left[ u - \frac{1 - 4N^2}{4N} \right] \quad (2.30)$$

This is the equation of a parabola centered about the point  $\frac{1}{4N}$  on the frequency axis. The conic opens in the direction of negative  $\underline{u}$ , its vertex lying at the value  $\underline{u}_{\max}$  given by (2.21). The distance between focus and vertex is  $\frac{1}{16N}$ .

Thus, the four impedance functions of a crystal are shown to be dependent only on the intrinsic crystal resistance  $\underline{R}_s$ ; on the normalized capacitance ratio  $\underline{N}$ ; and, through the parametric variable  $\underline{u}$ , on the nor-

malized frequency deviation  $D$ . Since  $\underline{N}$  may be varied arbitrarily from the minimum value  $\frac{n_0}{Q_s}$  to any desired magnitude, simply by increasing the external shunt capacitance  $\underline{C}_x$ , it is evident that a given crystal does not have a unique characteristic from the point of view adopted here. In Chapter **III**, the dependence of various features of the crystal impedance-frequency characteristics on the parameter  $\underline{N}$  is examined with the ultimate purpose of determining optimum specifications of this parameter for different applications.

## 2-8. The Direct Impedance Functions of a Crystal

Instead of substituting for  $\underline{D}$ , the expression (2.25) for  $\underline{u}$  may be inserted in (2.11) to give, with (2.1):

$$R(w) = \frac{R_s}{(1 - 2ND)^2 + N^2} \quad (2.31)$$

The formulas for the other impedance functions containing  $\underline{D}$  explicitly are, however, quite cumbersome.

It is evident from (2.27) and (2.28) that  $X(w)$  and  $|Z(w)|$  are derivable from the resistance function through multiplication by a function of  $\underline{u}$ . The nature of the  $\underline{u}$  functions involved will be worth examining, as from them we may deduce the reactance and impedance-magnitude functions knowing the form of (2.31).

Figure 5 shows a family of curves of crystal resistance function versus frequency. The equivalent series resistance  $R(w)$  has been referred to the basic impedance level of the crystal,  $\underline{R}_s$ , and the normalized frequency deviation  $\underline{D}$  is expressed as a fraction of the value  $\frac{1}{2N}$ .

In Figure 6, the reactance factor  $\underline{u}$  has been plotted from (2.25)

for values of  $\underline{N}$  corresponding to those in Figure 5. The reactance function can thus be obtained by multiplying the value of the resistance function by the associated value of  $\underline{u}$  at every point on the frequency axis.

The impedance-magnitude factor  $\sqrt{1 + u^2}$  is shown in Figure 7. Multiplying  $R(w)$  by the value of this factor at every point  $\underline{D}$  on corresponding  $\underline{N}$  curves yields the impedance-magnitude function versus frequency.

Finally, Figure 8 gives the phase angle function  $\theta = \arctan u$ .

The frequency scale in all four figures deserves clarification. It will be noted that the variable employed is  $2ND$ —that is, the frequency scale is different for each curve plotted. From the definitions of Chapter I, the following equivalences may be written:

$$2ND = 2nd = 2n(c - 1) = \frac{2n}{w_s}(w - w_s) \quad (2.32)$$

Thus, the actual frequency represented by  $2ND$  involves the reference frequency  $\underline{w}_s$  and the capacitance ratio  $\underline{n}$ .

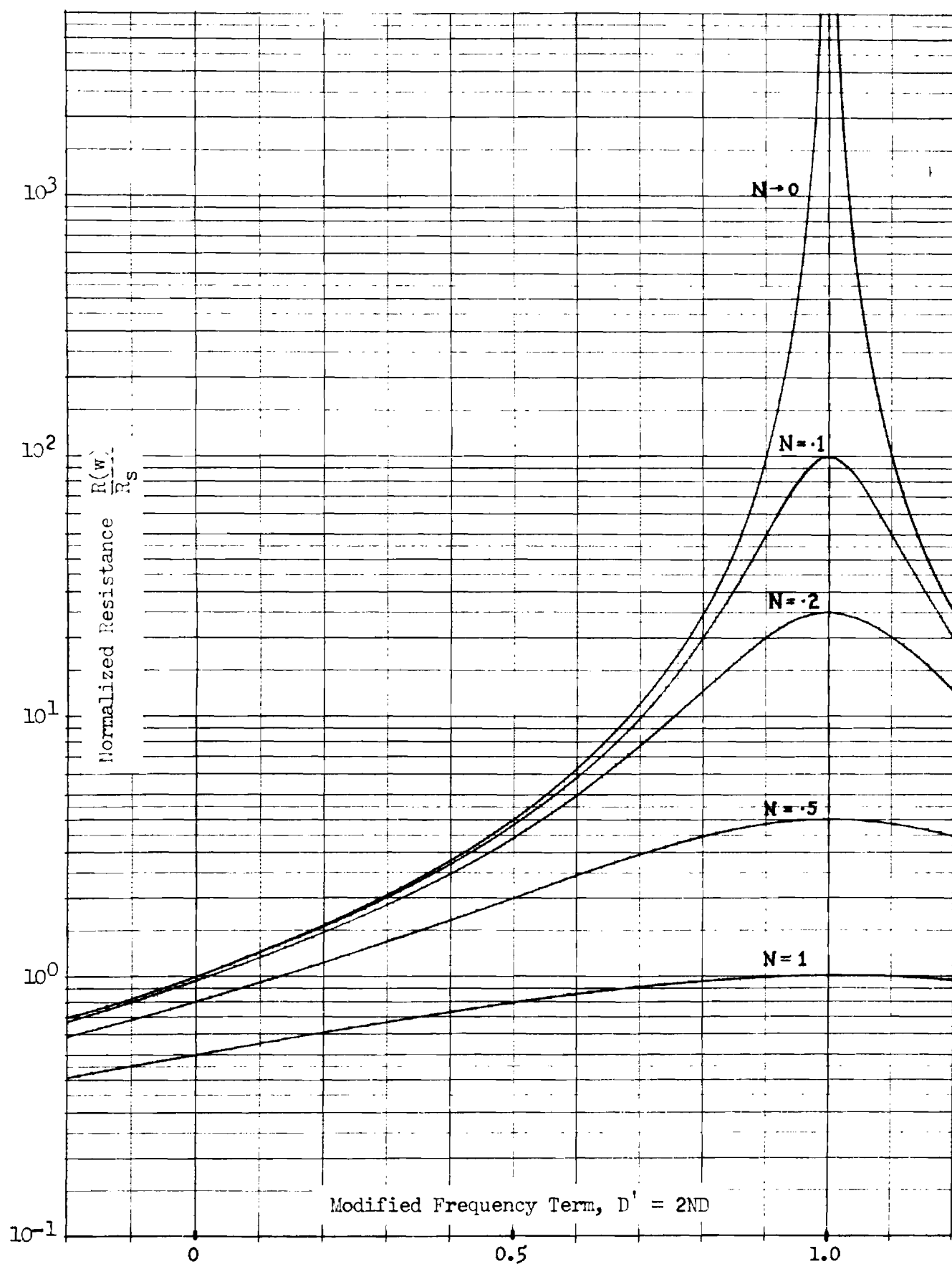


Figure 5. Normalized Crystal Resistance Function Versus Modified Normalized Frequency Deviation, for Various Values of  $N$ .

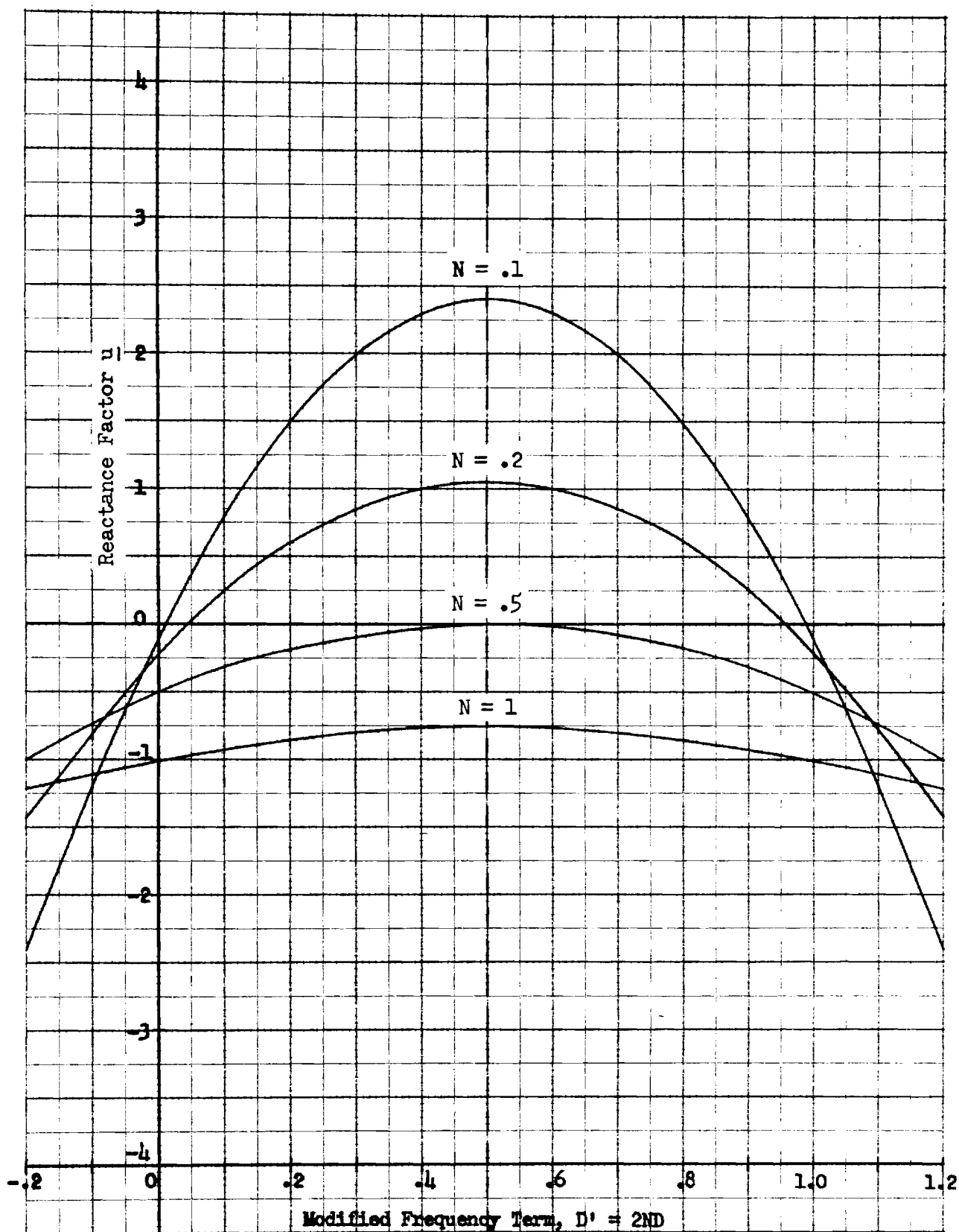


Figure 6. Reactance Factor Versus Modified Normalized Frequency Deviation, for Various Values of  $N$ .

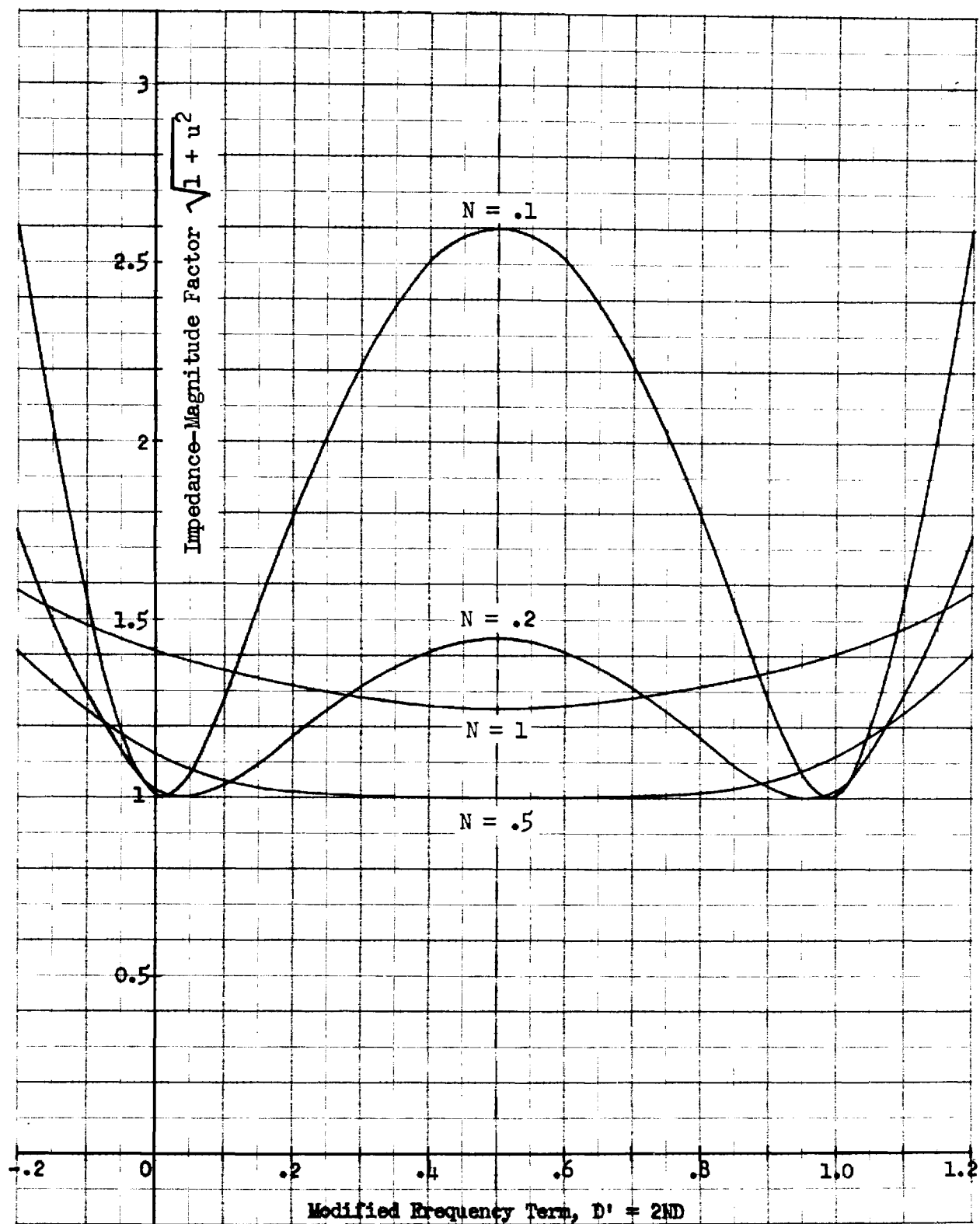


Figure 7. Impedance-Magnitude Factor Versus Modified Normalized Frequency Deviation, for Various Values of  $N$ .

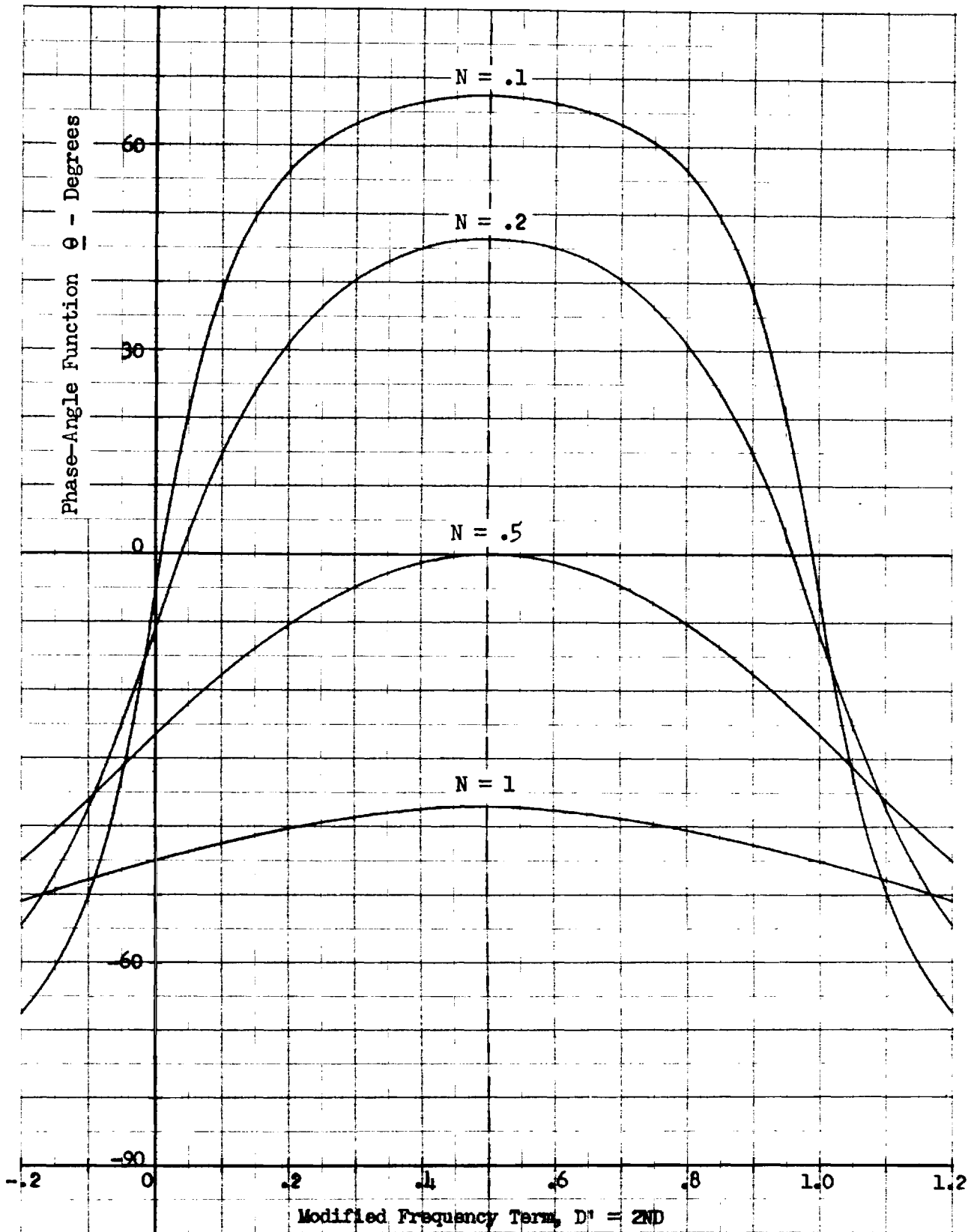


Figure 8. Phase-Angle Function Versus Modified Normalized Frequency Deviation, for Various Values of  $N$ .

### III. RAPID PLOTTING OF THE CRYSTAL IMPEDANCE FUNCTIONS

Figures 9(A) and 9(B) show plots of the four impedance functions of a crystal when the normalized capacitance ratio has the value 0.2—that is, when the capacitance ratio of the crystal with the associated external  $C_x$  is 0.2 times as large as the quality factor  $Q_s$  of the series RLC arm. These curves were obtained in the manner outlined under Section 2-8. The impedance levels have been referred to the basic value  $R_s$ , and the frequency denoted by the abscissa depends on the actual value of  $Q_s$  according to Equation (1.10).

As is evident from the families of curves given in Figures 5 through 8 of the preceding chapter, the general shape of the impedance functions is relatively unaffected by the characterizing crystal constant  $N$ . Consequently, if a few critical points on the impedance plots could be determined for every value of  $N$ , it would be possible to sketch the crystal characteristics quite accurately, without resort to the parametric equations of Section 2-7 or to the factor curves of Section 2-8.

A number of mathematically salient points are now investigated for dependency upon  $N$ . Curves are plotted to express the relationships more conveniently. And, finally, the method of use is illustrated by drawing a set of characteristics, similar to those of Figure 9, for a specific crystal.

#### 3-1. Resistance Maximum

The maximum value of the equivalent crystal resistance is evidently defined by the general condition:



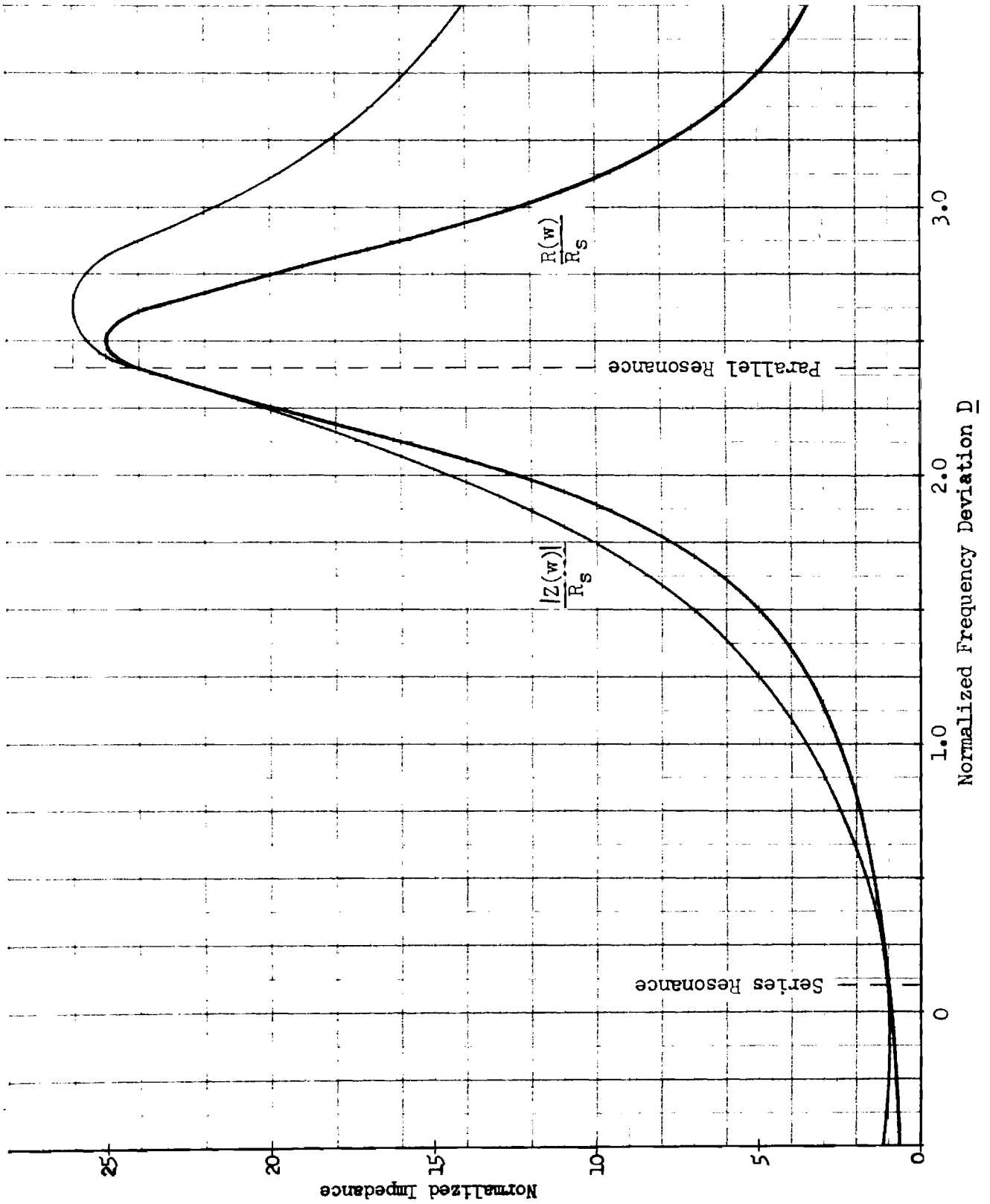


Figure 9 (A). Crystal Resistance and Impedance-Magnitude Functions Versus Normalized Frequency Deviation, for  $N = 0.2$ .

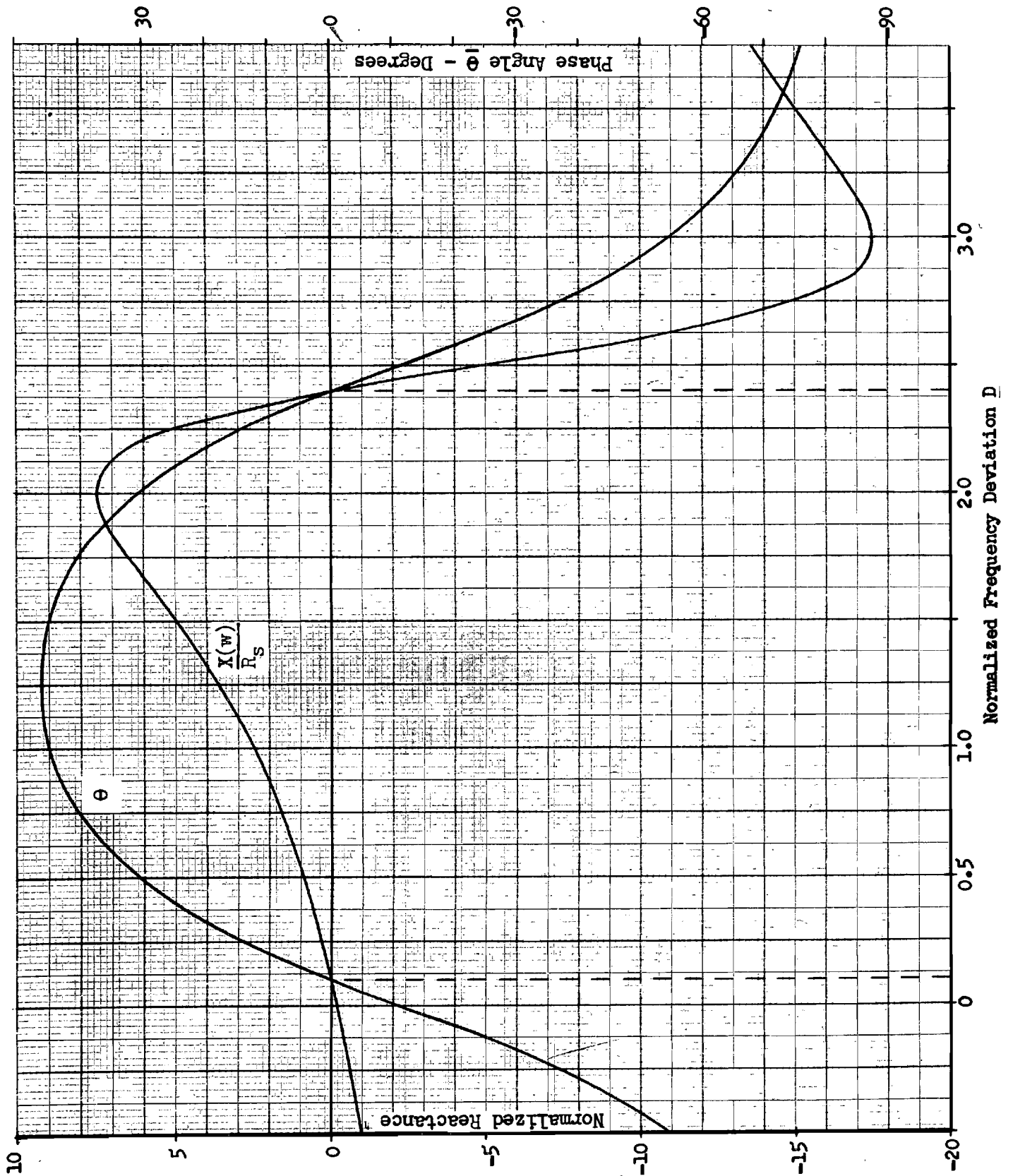


Figure 9 (B). Crystal Reactance and Phase-Angle Functions  
Versus Normalized Frequency Deviation, for  $N = 0.2$ .

$$\frac{dR(w)}{dD} = 0 \quad (3.1)^9$$

The explicit condition is easily found from (2.31) to be:

$$D_{R_{\max}} = \frac{1}{2N} \quad (3.2)$$

Substituting (3.2) into (2.31), there results:

$$\frac{R(w)_{\max}}{R_s} = \frac{1}{N^2} \quad (3.3)$$

And, from (2.25),

$$u_{R_{\max}} = -N \quad (3.4)$$

The value of the reactance function when  $R(w)$  is a maximum is determined from (2.27) in conjunction with (3.3) and (3.4):

$$\frac{X(w)_{R_{\max}}}{R_s} = -\frac{1}{N} \quad (3.5)$$

The values of the impedance-magnitude and phase angle functions also are easily found, from (2.28) and (2.29).

### 3-2. Series and Parallel Resonance

A full discussion of the series and parallel resonant frequencies of a crystal is presented in Appendix A. These frequencies are denoted by the symbols  $\underline{w}_r$  and  $\underline{w}_a$ , respectively ( $\underline{r}$  for resonance,  $\underline{a}$  for antiresonance); and the subscripts are now employed to refer to the two conditions defined by zero phase angle.

---

<sup>9</sup>Because of typographical difficulties, the symbol  $\frac{dF}{dx}$  will denote both total and partial derivatives of a function  $\underline{F}$  with respect to a variable  $\underline{x}$ .

The resonant values of equivalent crystal resistance are most readily obtained by setting  $u = 0$  in (2.26):

$$\frac{R(w_r)}{R_s} = \frac{2}{1 + \sqrt{1 - 4N^2}} \quad (3.6a)$$

and,

$$\frac{R(w_a)}{R_s} = \frac{2}{1 - \sqrt{1 - 4N^2}} \quad (3.7a)$$

From (2.24), the values of  $\underline{D}$  corresponding to these resonant resistances are:

$$D_r = \frac{1 - \sqrt{1 - 4N^2}}{4N} \quad (3.8a)$$

and,

$$D_a = \frac{1 + \sqrt{1 - 4N^2}}{4N}$$

It is interesting to note that when  $\underline{N}$  is very small the expression  $\sqrt{1 - 4N^2}$  is approximately equal to  $(1 - 2N^2)$ , and Equations (3.6a) through (3.9a) become:<sup>10</sup>

$$\frac{R(w_r)}{R_s} = 1 \quad (3.6b)$$

$$\frac{R(w_a)}{R_s} = \frac{1}{N^2} \quad (3.7b)$$

$$D_r = \frac{N}{2} \quad (3.8b)$$

$$D_a = \frac{1}{2N} \quad (3.9b)$$

---

<sup>10</sup>By the Binomial Theorem:  $(1 - x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} + \dots$ . When  $x$  is small compared to unity, the second and higher degree terms of the series may be neglected.

This second group of equations would apply quite accurately for values of  $\underline{N}$  as large as 0.1. Equation (3.6b) shows the validity of the common assumption that the equivalent series resistance of a crystal is the same as its intrinsic resistance  $\underline{R}_s$  in the vicinity of series resonance, provided the capacitance ratio  $\underline{n}$  is not too large compared to the crystal  $\underline{Q}$  (i.e.,  $\underline{N}$  is small). Equation (3.7b) indicates that, for  $\underline{N}$  sufficiently small, the resistance at antiresonance is essentially the same as the maximum resistance, given by (3.3).

The "performance index" (PI) of a crystal has been defined by Harrison as "the anti-resonant resistance of the crystal and holder having in parallel with it the capacitance introduced by the remainder of the oscillator."<sup>11</sup> Since slightly different definitions of the PI also exist, it is deemed inadvisable to correlate specifically either of the equations (3.7a) or (3.7b) with this new crystal quantity.

### 3-3. Double Resistance Root

When the discriminant (2.22) is zero, the frequency term  $\underline{d}$  is single-valued. This occurs for

$$\underline{u} = \frac{1 - \underline{L}\underline{N}^2}{\underline{L}\underline{N}} \quad (3.10)$$

Substituting the above value into (2.26), and denoting the frequency corresponding to (3.10) by  $\underline{w}_{12}$ , there results:

$$\frac{R(\underline{w}_{12})}{\underline{R}_s} = \frac{\underline{L}}{1 + \underline{L}\underline{N}^2} \quad (3.11)$$

---

<sup>11</sup>R. A. Heising, Quartz Crystals for Electrical Circuits (New York: D. Van Nostrand Company, 1946), p. 461.

From (2.24), the associated normalized frequency term is:

$$D_{12} = \frac{1}{4N} \quad (3.12)$$

It should be noted that (3.10) was obtained in the same manner as (2.21), and the two express the same relationship between  $u$  and  $N$ . When  $N$  has the value 0.5, the phase angle becomes zero, which means that the series and parallel resonant frequencies have coalesced. (This fact may be verified by reference to (3.8a) and (3.9a).) It follows that the crystal cannot ever become inductive for  $N \geq 0.5$ . This statement is of considerable physical importance in that inductive reactance is essential in the most widely used oscillator circuits.

### 3-4. Reactance Maximum and Minimum

Possible maxima and minima of reactance occur when

$$\frac{dX(w)}{dD} = 0 \quad (3.13)$$

By the chain rule for derivatives this becomes:

$$\frac{dX(w)}{du} \cdot \frac{du}{dD} = 0 \quad (3.14)$$

Using  $X(w)$  as given by (2.27) and performing the indicated operations on (2.25) and (2.26), there results finally:

$$\frac{\mp 4 \left[ \sqrt{1 - 4N(N+u)} \mp (1 - 4N^2 - 2Nu) \right]}{\left[ 1 - 2Nu \mp \sqrt{1 - 4N(N+u)} \right]^2} = 0 \quad (3.15)$$

Upon squaring the numerator, a quadratic in  $u$  is obtained:

$$\frac{u^2}{2} + 2Nu + \frac{(4N^2 - 1)}{2} = 0 \quad (3.16)$$

The solutions are:

$$u = - 2N \pm 1 \quad (3.17)$$

Since the resistance of the crystal is always positive, it is easy to see that the upper (positive) sign of (3.17) denotes a maximum of reactance, and the lower (negative) a minimum.

It is interesting to note that the maximum and minimum of reactance do not occur at phase angles of exactly  $\pm 45$  degrees ( $u = \tan \theta = 1$ ) as is usually assumed. However, for  $N$  very small such an approximation is good.

The frequency terms corresponding to the limiting values of reactance are obtained by substituting (3.17) in (2.24):

$$D_{X_{\max}} = \frac{1 - N}{2N} \quad (3.18)$$

$$D_{X_{\min}} = \frac{1 + N}{2N} \quad (3.19)$$

The values of the resistance function prove to be equal at the points defined by (3.18) and (3.19). Substituting (3.18) into (2.26),

$$\frac{R(w_{\frac{1}{2}})}{R_s} = \frac{1}{2N^2} \quad (3.20)$$

The subscript " $\frac{1}{2}$ " is used to denote this point because the resistance value is one-half its maximum value.

The actual maximum and minimum of reactance are obtained from substituting (3.20) and (3.18) in (2.27):

$$\frac{X(w)_{\max}}{R_s} = \frac{1 - 2N}{2N^2} \quad (3.21)$$

$$\frac{X(w)_{\min}}{R_s} = - \frac{(1 + 2N)}{2N^2} \quad (3.22)$$

### 3-5. Tabulation of Critical-Point Functions

Many more special values of the equivalent series resistance, reactance, and impedance of a crystal may be determined as functions of  $N$  by manipulations similar to those exhibited above. A list of some of these  $N$ -functional forms is given in Table I, placed in order of increasing frequency. Graphed presentations are included as Figures 10 through 17. Table II shows the approximate expressions which result when  $N$  is small—say, less than 0.1—permitting some terms in  $N^2$  to be neglected.

A number of interesting facts are brought out by the information given in Table I. For example, the difference between the maximum and minimum values of crystal reactance is always equal to the maximum value of crystal resistance; as is, also, the difference between the maximum and minimum values of impedance. In equation form,

$$X_{\max} - X_{\min} = |Z|_{\max} - |Z|_{\min} = R_{\max} = \frac{R_s}{N^2} \quad (3.23)$$

The minimum value of reactance is always greater in magnitude than the maximum value, by an amount:

$$|X_{\min}| - |X_{\max}| = \frac{2R_s}{N} \quad (3.24)$$

The frequency separation of these extreme values of reactance is:

$$\Delta D = D_{X_{\min}} - D_{X_{\max}} = 1 \quad (3.25)$$



TABLE I. LIST OF CRITICAL-POINT N-FUNCTIONAL FORMS

Ref. No.	Characteristic	D	u	$\frac{R(w)}{R_s}$	$\frac{X(w)}{R_s}$	$\frac{ Z(w) }{R_s}$	Slopes
I	$ Z _{\min}$	$\frac{1 - \sqrt{1 + 4N^2}}{4N}$	$-2N$	$\frac{2}{(1 + 4N^2) + \sqrt{1 + 4N^2}}$	$\frac{-4N}{(1 + 4N^2) + \sqrt{1 + 4N^2}}$	$\frac{2}{\sqrt{1 + 4N^2} + 1}$	
II	S.R.	$\frac{1 - \sqrt{1 - 4N^2}}{4N}$	0	$\frac{2}{1 + \sqrt{1 - 4N^2}}$	0	$\frac{2}{1 + \sqrt{1 - 4N^2}}$	$\frac{dX}{dD} = \frac{4\sqrt{1 - 4N^2}}{1 + \sqrt{1 - 4N^2}}$
III	$u_{\max}$	$\frac{1}{4N}$	$\frac{1 - 4N^2}{4N}$	$\frac{4}{1 + 4N^2}$	$\frac{1 - 4N^2}{N(1 + 4N^2)}$	$\frac{1}{N}$	$\frac{dX}{dD} = \frac{8(1 - 4N^2)}{(1 + 4N^2)^2}$
IV	$X_{\max}$	$\frac{1 - N}{2N}$	$1 - 2N$	$\frac{1}{2N^2}$	$\frac{1 - 2N}{2N^2}$	$\frac{\sqrt{1 + (1 - 2N)^2}}{2N^2}$	$\frac{d Z }{dD} = \frac{1}{N^2\sqrt{1 + (1 - 2N)^2}}$ $\frac{dR}{dD} = 1/N^2$
V	P.R.	$\frac{1 + \sqrt{1 - 4N^2}}{4N}$	0	$\frac{2}{1 - \sqrt{1 - 4N^2}}$	0	$\frac{2}{1 - \sqrt{1 - 4N^2}}$	$\frac{dX}{dD} = \frac{-4\sqrt{1 - 4N^2}}{1 - \sqrt{1 - 4N^2}}$
VI	$R_{\max}$	$\frac{1}{2N}$	$-N$	$\frac{1}{N^2}$	$-\frac{1}{N}$	$\frac{\sqrt{1 + N^2}}{N^2}$	
VII	$ Z _{\max}$	$\frac{1 + \sqrt{1 + 4N^2}}{4N}$	$-2N$	$\frac{2}{(1 + 4N^2) - \sqrt{1 + 4N^2}}$	$\frac{-4N}{(1 + 4N^2) - \sqrt{1 + 4N^2}}$	$\frac{2}{\sqrt{1 + 4N^2} - 1}$	
VIII	$X_{\min}$	$\frac{1 + N}{2N}$	$-1 - 2N$	$\frac{1}{2N^2}$	$-\frac{1 + 2N}{2N^2}$	$\frac{\sqrt{1 + (1 + 2N)^2}}{2N^2}$	$\frac{d Z }{dD} = \frac{-1}{N^2\sqrt{1 + (1 + 2N)^2}}$ $\frac{dR}{dD} = -1/N^2$

Ref. No.	D	u	$\frac{R(w)}{R_s}$	$\frac{X(w)}{R_s}$	$\frac{ Z(w) }{R_s}$	Slopes
I	$-\frac{N}{2}$	$-2N$	1	$-2N$	1	
II	$\frac{N}{2}$	0	1	0	1	$\frac{dX}{dD} = 2$
III	$\frac{1}{4N}$	$\frac{1}{4N}$	4	$\frac{1}{N}$	$\frac{1}{N}$	$\frac{dX}{dD} = 8$
IV	$\frac{1-N}{2N}$	$1-2N$	$\frac{1}{2N^2}$	$\frac{1-2N}{2N^2}$	$\frac{1-N}{\sqrt{2} N^2}$	$\frac{d Z }{dD} = \frac{1}{\sqrt{2} N^2}$ $\frac{dR}{dD} = 1/N^2$
V	$\frac{1}{2N}$	0	$\frac{1}{N^2}$	0	$\frac{1}{N^2}$	$\frac{dX}{dD} = \frac{-2}{N^2}$
VI	$\frac{1}{2N}$	$-N$	$\frac{1}{N^2}$	$-\frac{1}{N}$	$\frac{1}{N^2}$	
VII	$\frac{1}{2N}$	$-2N$	$\frac{1}{N^2}$	$-\frac{2}{N}$	$\frac{1}{N^2}$	
VIII	$\frac{1+N}{2N}$	$-1-2N$	$\frac{1}{2N^2}$	$-\frac{1+2N}{2N^2}$	$\frac{1+N}{\sqrt{2} N^2}$	$\frac{d Z }{dD} = \frac{-1}{\sqrt{2} N^2}$ $\frac{dR}{dD} = -1/N^2$

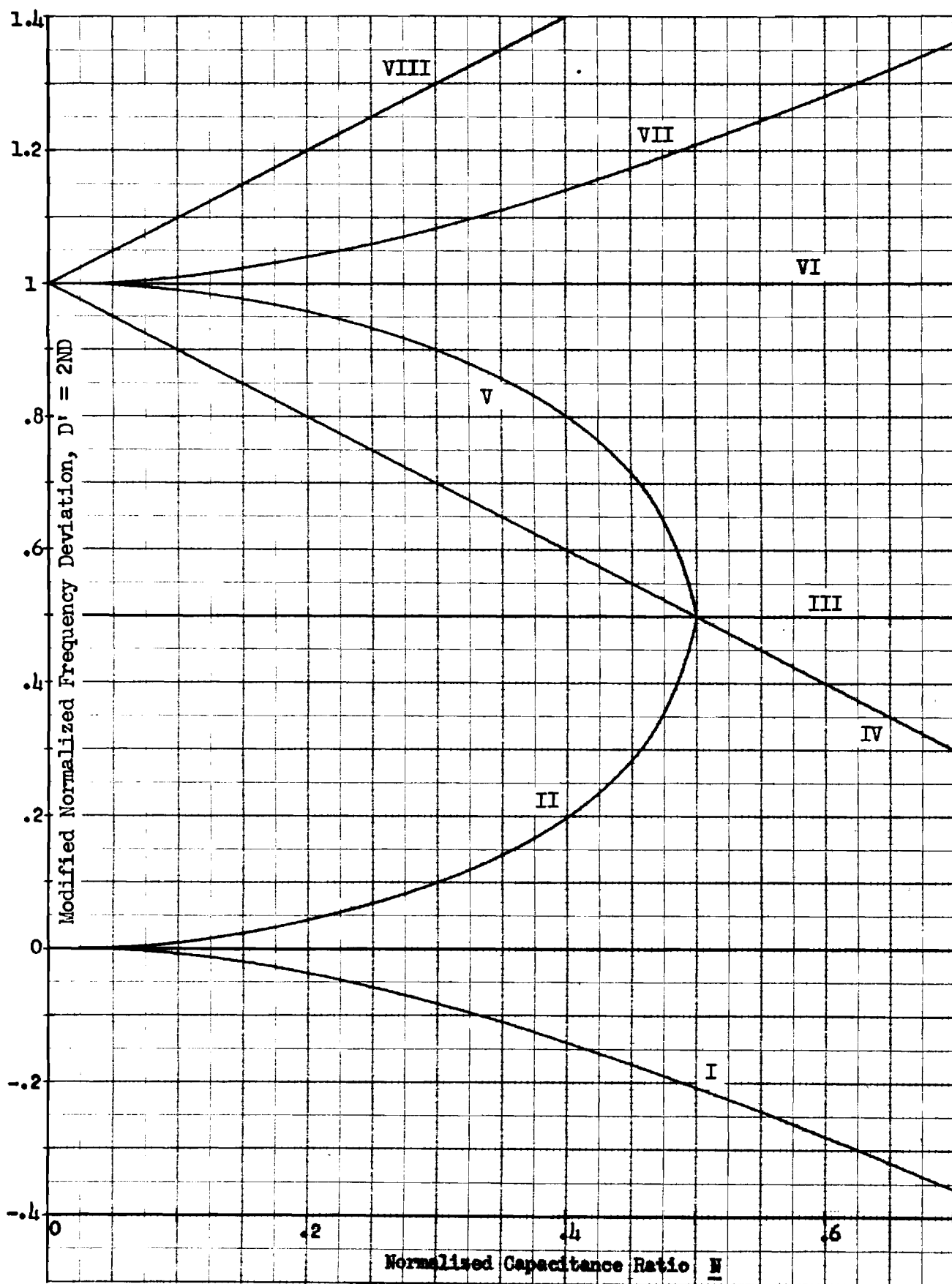


Figure 10. Critical-Point Frequency Terms Versus Normalized Capacitance Ratio.

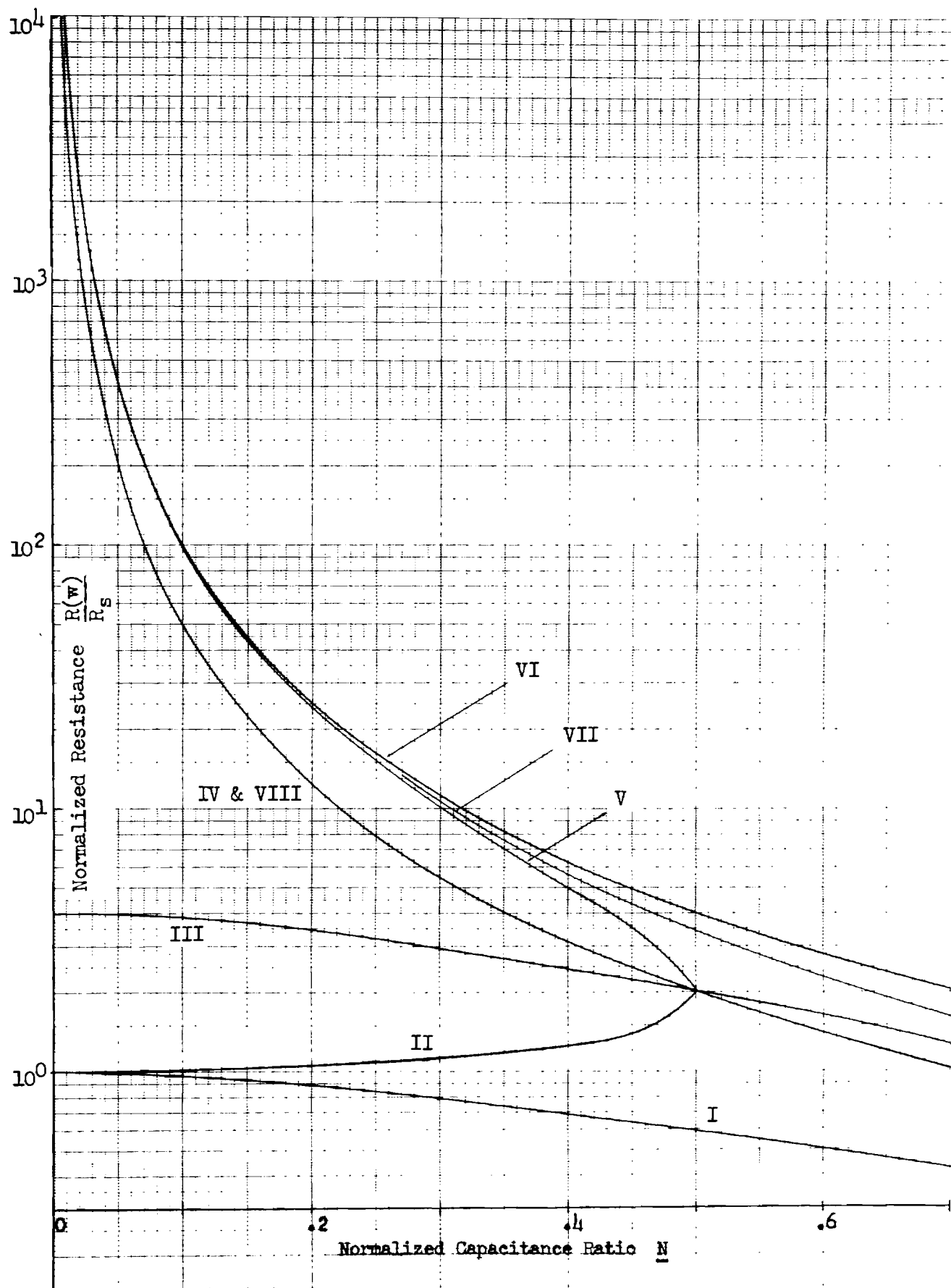


Figure 11. Critical-Point Normalized Resistance Versus Normalized Capacitance Ratio.

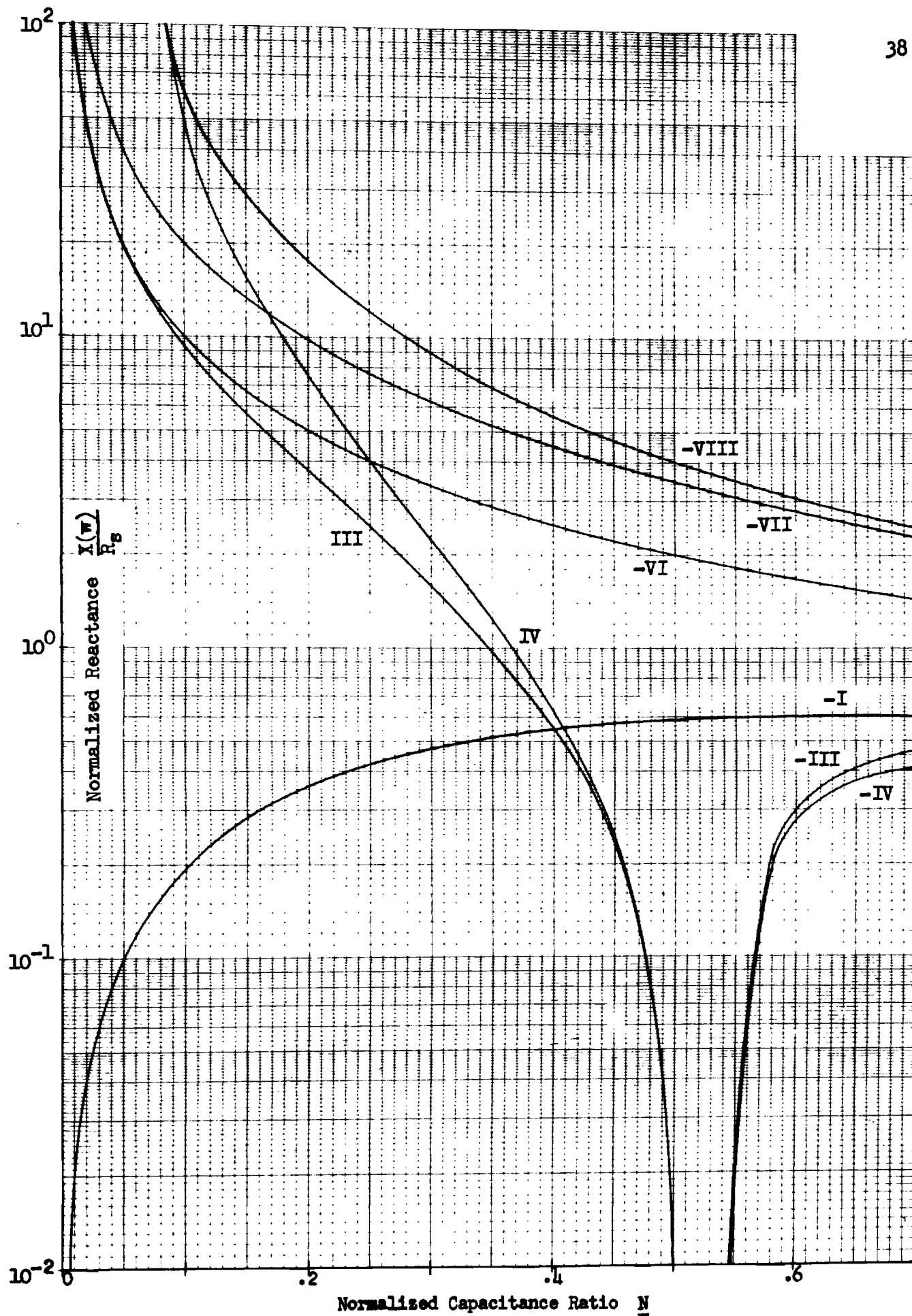


Figure 12. Critical-Point Normalized Reactance Versus Normalized Capacitance Ratio.

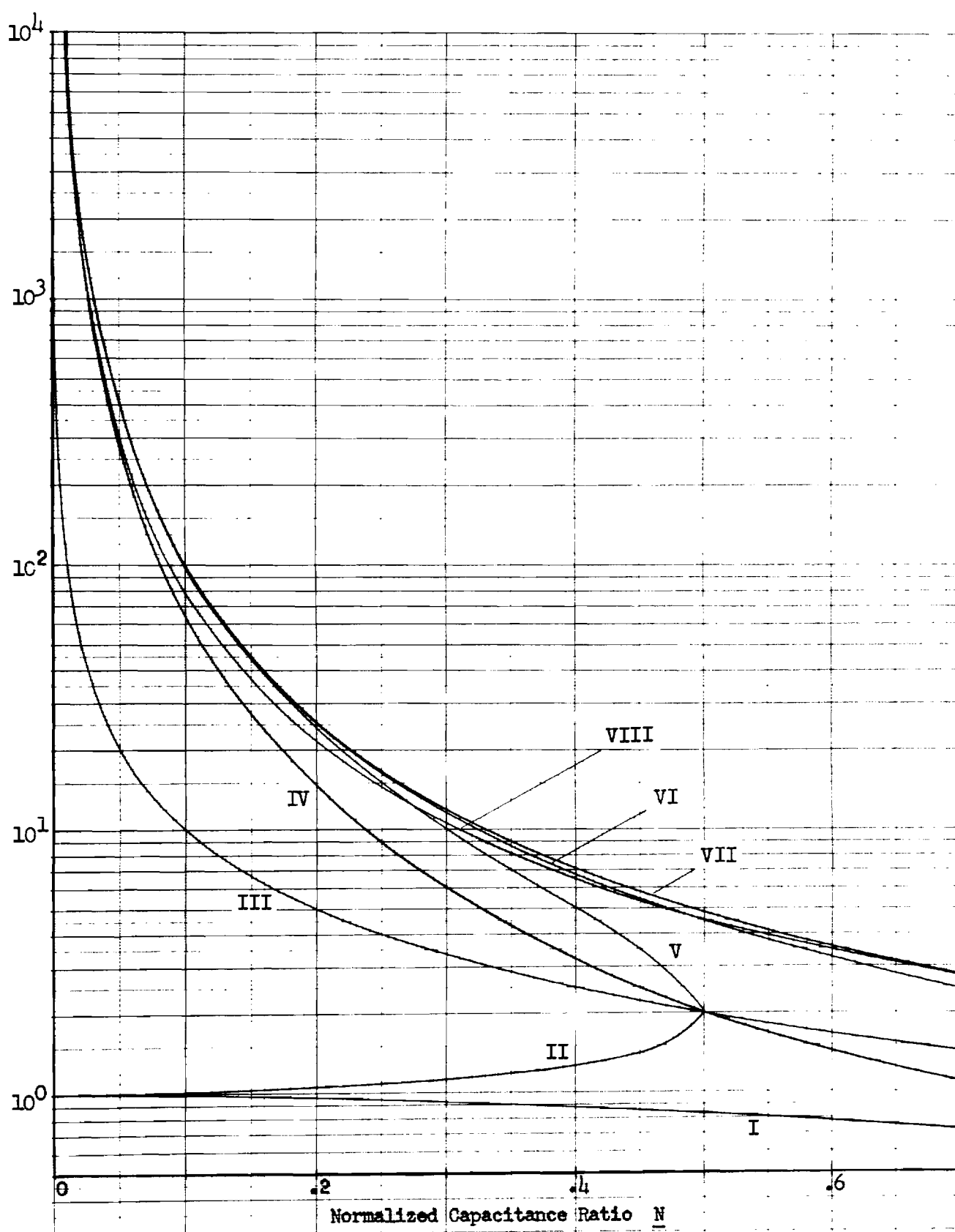


Figure 13. Critical Point Normalized Impedance Magnitude Versus Normalized Capacitance Ratio.

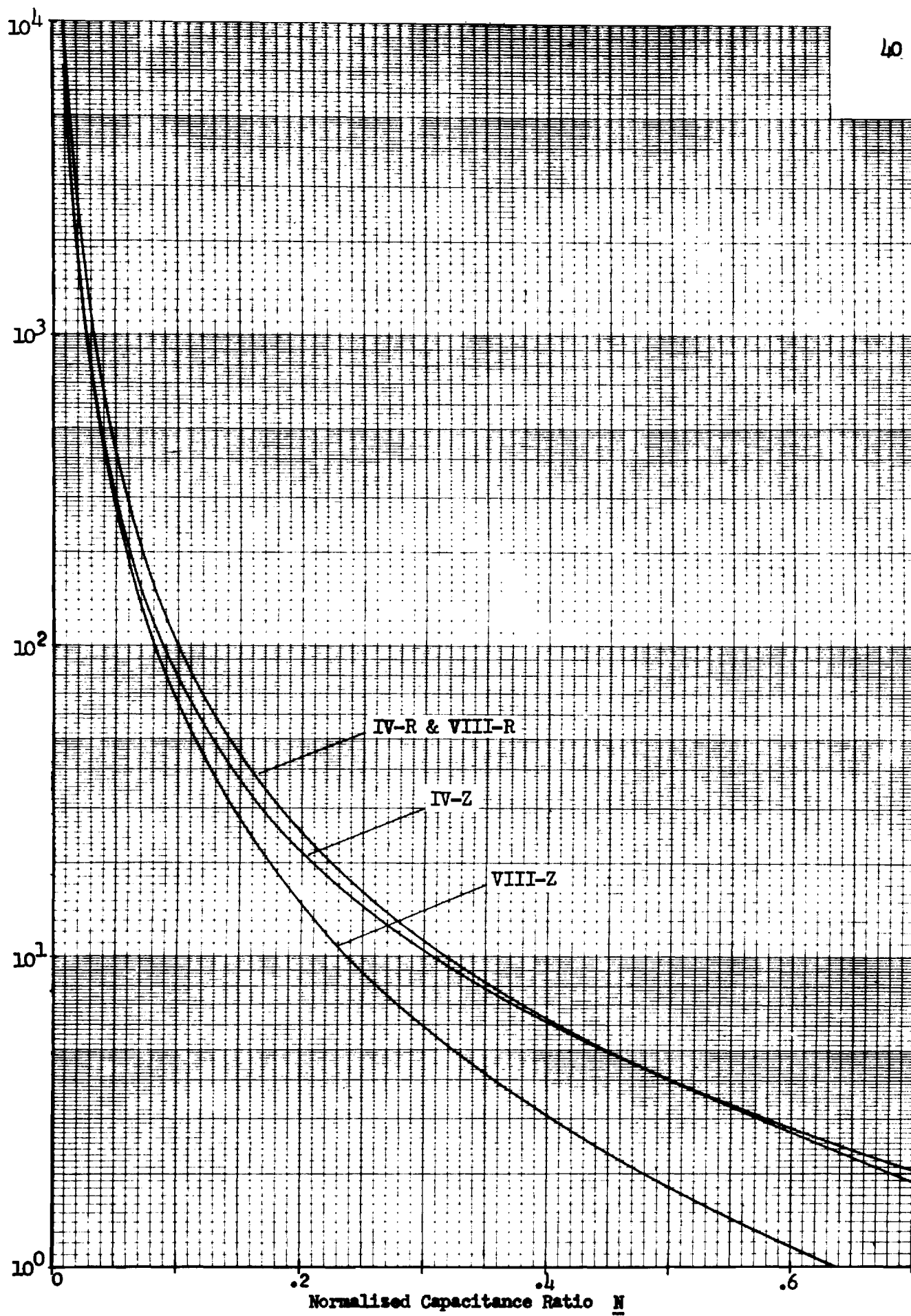


Figure 14. Critical-Point Normalized Resistance and Impedance-Magnitude Derivatives, Versus Normalized Capacitance Ratio.

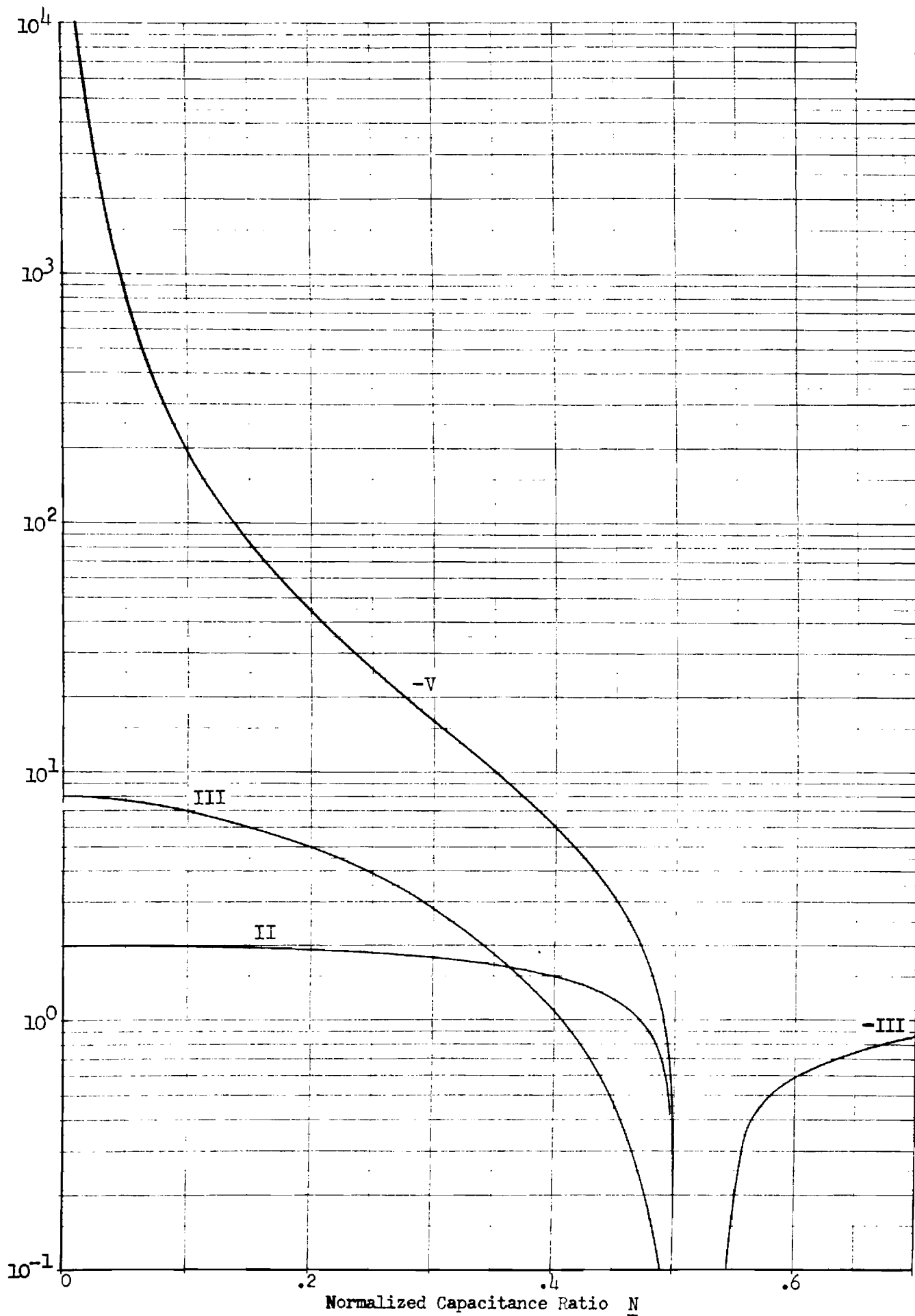


Figure 15. Critical-Point Normalized Reactance Derivatives, Versus Normalized Capacitance Ratio.



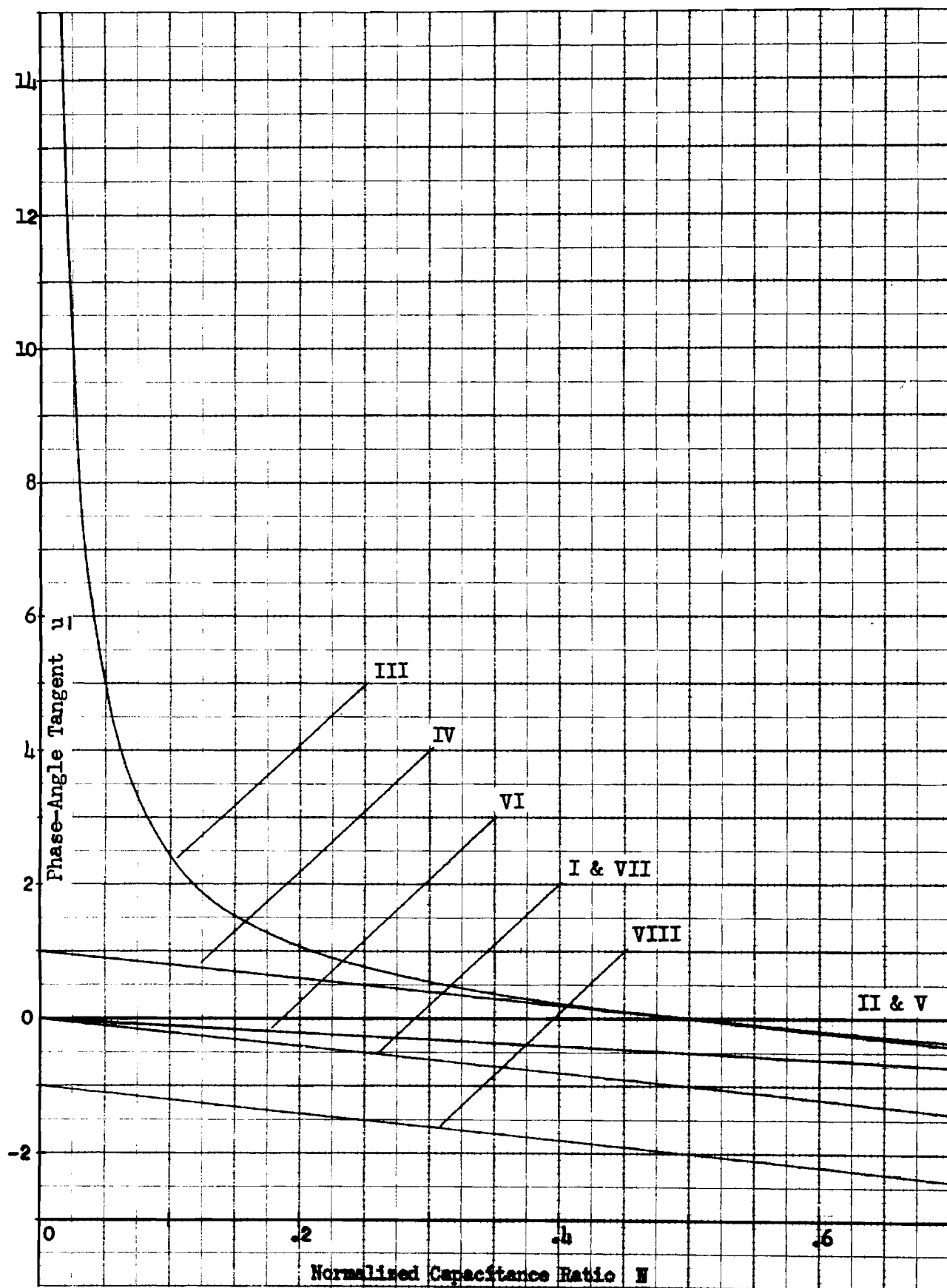


Figure 16. Critical-Point Phase-Angle Tangent Versus Normalized Capacitance Ratio.

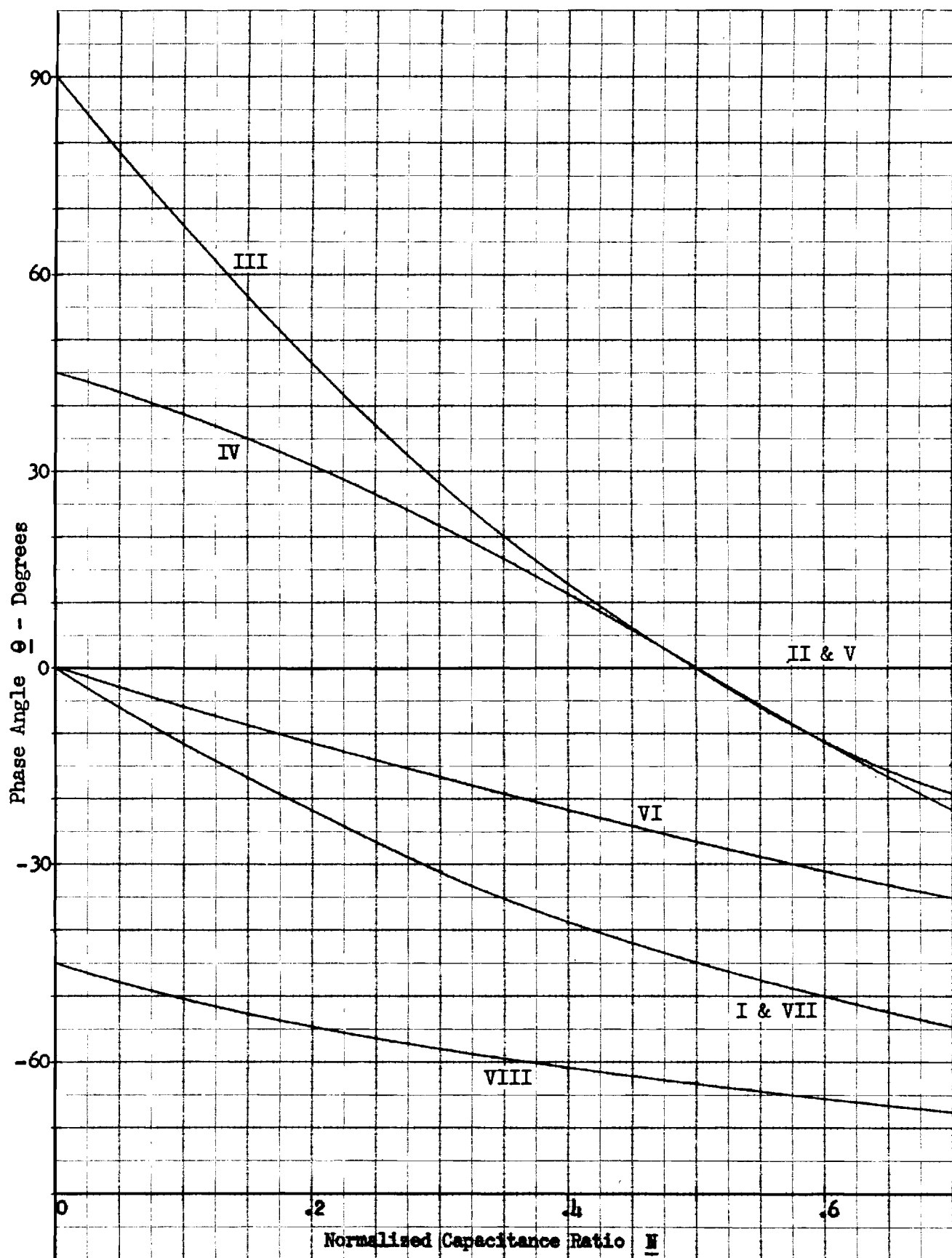


Figure 17. Critical-Point Phase Angle Versus Normalized Capacitance Ratio.

The un-normalized form of (3.25)—namely,

$$\Delta d = \frac{1}{Q_s} \quad (3.26)$$

will be recognized as the "half-power band width" of a high- $Q$  resonant circuit. This result was to be expected since the frequencies of the maximum and minimum of reactance are the same as those at which the resistance function has one-half its maximum value.

From Table I we may also find the separation of the resonances to be:

$$D_a - D_r = \frac{\sqrt{1 - 4N^2}}{2N} \quad (3.27)$$

From the approximate forms for small  $N$  is obtained the familiar expression:

$$d_a - d_r \doteq \frac{1}{2n} \quad (3.28)$$

Thus, whenever the capacitance ratio  $n$  of a crystal is small compared with its quality factor  $Q_s$ , the resonances are separated by a frequency band whose width is essentially inversely proportional to the capacitance ratio.

A final item of interest is that the slope of the reactance curve at antiresonance is always greater in magnitude than that at resonance, as evidenced by the ratio:

$$\begin{aligned} \frac{\left(\frac{dX}{dD}\right)_a}{\left(\frac{dX}{dD}\right)_r} &= \frac{1 + \sqrt{1 - 4N^2}}{1 - \sqrt{1 - 4N^2}} \\ &\doteq \frac{1}{N^2} \text{ for } N \text{ small.} \end{aligned} \quad (3.29)$$

### 3-6. Method of Plotting Crystal Impedance Characteristics

The curves of Figures 10 through 17 have been numbered with reference to the list of critical points given in Tables I and II. There are eight separate values of the frequency variable  $\underline{D}$ , each a function of  $\underline{N}$ , as shown in Figure 10. The resistance characteristic of a crystal may be determined by plotting the eight values of  $\underline{R}$  given in Figure 11 against the values of  $\underline{D}$  from Figure 10, using the  $\underline{N}$  of the given crystal. The slopes of the resistance curve at one or two of these points are also of assistance in sketching the curve; resistance derivatives have been plotted in Figure 14.

In the same way, the reactance function may be drawn by means of Figures 10, 12, and 15; the impedance-magnitude function by Figures 10, 13, and 14; and the phase-angle function by Figures 10 and 17. Figure 16 applies to the phase-angle tangent, which is not usually plotted but is serviceable in certain problems of analysis.

As an example, we shall sketch the impedance-frequency characteristics of a crystal having the following measured or calculated constants:

$$\begin{aligned} R_s &= 100 \text{ ohms} \\ L_s &= 10 \text{ henries} \\ C_s &= 0.1 \text{ micromicrofarad} \\ C_o &= 10 \text{ micromicrofarads} \end{aligned}$$

Assume a circuital shunt capacitance  $C_x = 990$  micromicrofarads. Then, in accordance with the definitions of Chapter I, we find that:

$$\begin{aligned} \omega_s &= 1,000,000 \text{ rad/sec} \quad (159.2 \text{ kilocycles}) \\ Q_s &= 100,000 \end{aligned}$$

$$\begin{aligned} n &= 10,000 & (n_0 &= 100) \\ N &= 0.1 \end{aligned}$$

The above numbers are typical<sup>12</sup> except that  $C_x$  will ordinarily be closer to one-tenth the value chosen.

The next step is to draw a vertical line at  $N = 0.1$  on each of the Figures, 10 through 17. The coordinates of points on the four impedance functions are then read from corresponding curves in appropriate figures. Taking into account the normalizing factors, the final results are obtained in dimensional units, as shown by Figures 18(A) and 18(B).

---

<sup>12</sup>W. G. Cady, Piezoelectricity (New York: McGraw-Hill Book Company, 1946), p. 461.

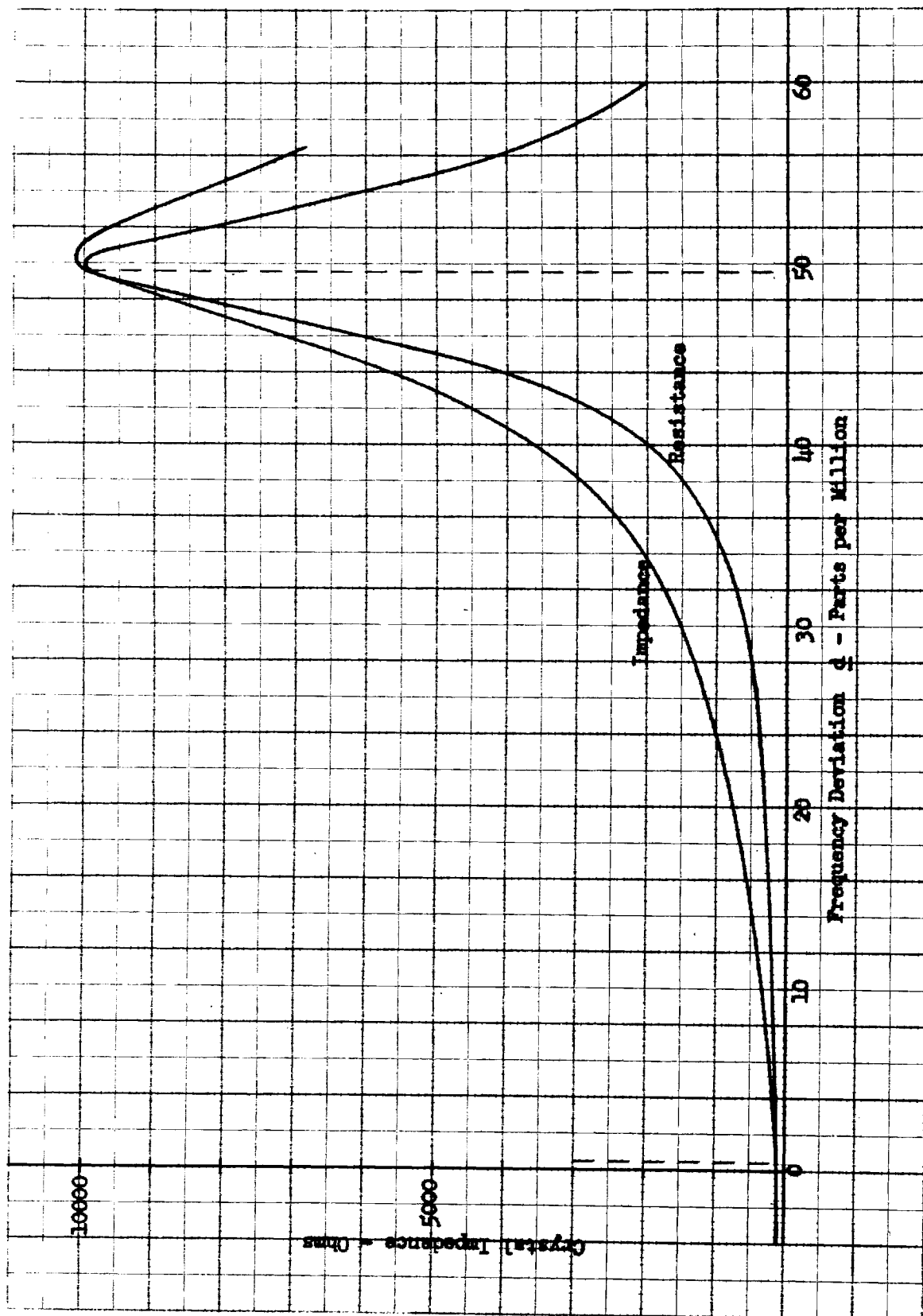


Figure 18(A). Resistance and Impedance-Magnitude Functions for a Crystal Having a Normalized Capacitance Ratio of  $N = 0.10$ .

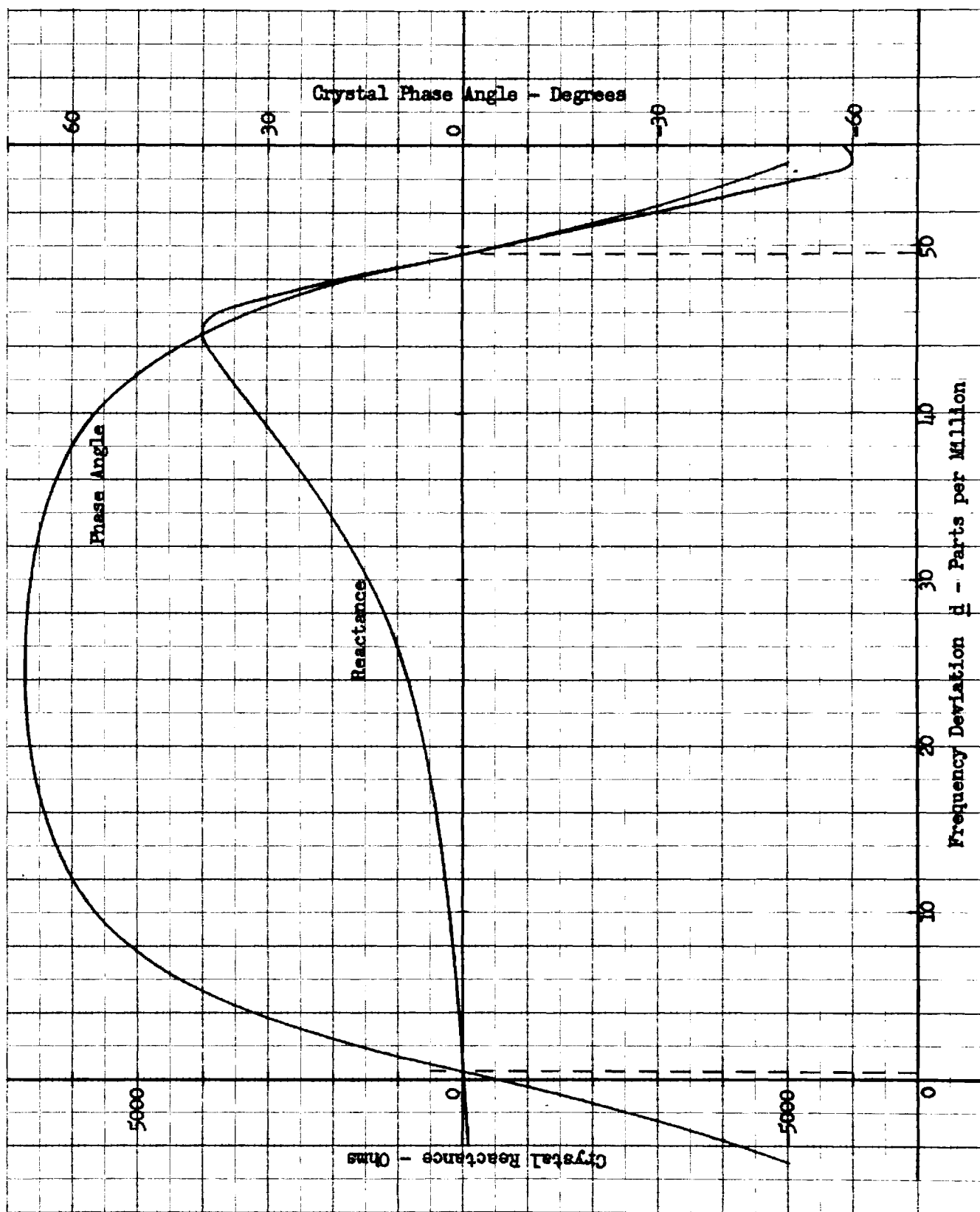


Figure 18(B). Reactance and Phase-Angle Functions for a Crystal Having a Normalized Capacitance Ratio of  $N = 0.10$ .

#### IV. THE EQUIVALENT ADMITTANCE OF A CRYSTAL

The analysis of the two preceding chapters can be repeated to obtain the admittance functions of a crystal as introduced in Section 1-7. The forms of the admittance functions may also be arrived at from manipulations of the impedance functions already derived.

##### 4-1. The Admittance Functions of a Crystal

We first write:

$$Y(w) = G(w) + jB(w) = \frac{1}{Z(w)} = \frac{1}{R(w) + jX(w)} = \frac{1}{R(w)(1 + ju)} \quad (4.1)$$

Rationalizing, and separating reals and imaginaries,

$$G(w) = \frac{1}{R(w)(1 + u^2)} = \frac{R(w)}{|Z(w)|^2} \quad (4.2)$$

$$B(w) = \frac{-u}{R(w)(1 + u^2)} = \frac{-X(w)}{|Z(w)|^2} \quad (4.3)$$

Since the impedance of a crystal can be expressed in the form,

$$Z(w) = |Z(w)|e^{j\theta} \quad (4.4)$$

the definition of admittance enables us to write:

$$Y(w) = \frac{e^{-j\theta}}{|Z(w)|} = |Y(w)| \angle -\theta \quad (4.5)$$

where

$$|Y(w)| = \frac{1}{|Z(w)|} = \frac{1}{R(w) \cdot \sqrt{1 + u^2}} \quad (4.6)$$

and, as before,

$$\theta = \arctan u \quad (4.7)$$



Thus, the four admittance functions  $G(\omega)$ ,  $B(\omega)$ ,  $|Y(\omega)|$ , and  $-\theta$  can be plotted versus frequency when  $R(\omega)$  and  $\underline{u}$  are given as functions of frequency.

#### 4-2. Crystal Conductance and Susceptance

To obtain further insight into the nature of the electrical characteristics of a crystal, we shall investigate the curves which result when susceptance is plotted versus conductance. We may consider first the RLC branch of the crystal network.

From the discussion of Section 1-7, and using Equations (1.22) and (1.23), the conductance and susceptance functions of  $\underline{R}_S$ ,  $\underline{L}_S$ , and  $\underline{C}_S$  in series are:

$$G'(\omega) = \frac{R_S}{R_S^2 + X_S^2} \quad (4.8)$$

$$B'(\omega) = \frac{-X_S}{R_S^2 + X_S^2} \quad (4.9)$$

The quantity  $\underline{X}_S$  is given in terms of the crystal parameters by (1.12) and may be substituted into (4.8) and (4.9) to yield:

$$R_S \cdot G'(\omega) = \frac{1}{1 + x^2} \quad (4.10)$$

$$R_S \cdot B'(\omega) = \frac{-x}{1 + x^2} \quad (4.11)$$

$$= -x \cdot G'(\omega)$$

where

$$x = Q_S \left( c - \frac{1}{c} \right) \quad (4.12)$$

When the whole crystal circuit is considered, the only change in the form of the admittance components (4.10) and (4.11) is the addition

of a capacitive susceptance term to account for the parallel branch containing  $C_y$ —that is, we add the term  $\frac{-1}{X_y}$ . Using (1.11), the admittance functions now become:

$$R_s \cdot G(\omega) = R_s \cdot G'(\omega) \quad (4.13)$$

$$R_s \cdot B(\omega) = R_s \cdot B'(\omega) + Nc \quad (4.14)$$

#### 4-3. Polar Admittance Curve for the RLC Branch

Figure 19 shows Equation (4.11) plotted against (4.10) as the frequency  $c$  is made to change from zero to infinity. The result is a circle of unit diameter, having its center at the point  $R_s G = 0.5$ ,  $R_s B = 0$ . This curve is obtained regardless of what value is given to  $Q_s$ . However, the frequency scale along the circle will be different for different values of  $Q_s$ , as indicated by (4.12).

To obtain some idea of this frequency variation, one can determine its rate of change with respect to the angle  $\phi$  which the radius vector of the circle makes with the conductance axis. It is easy to show that:

$$\frac{dc}{d\phi} = \frac{Q_s^2(c^2 - 1)^2 + c^2}{4Q_s(c^2 + 1)} \quad (4.15)$$

At the points where the susceptance is zero, we have:

$$\left. \begin{aligned} \left( \frac{dc}{d\phi} \right)_{c=0} &= \frac{Q_s}{4} \\ \left( \frac{dc}{d\phi} \right)_{c=1} &= \frac{1}{8Q_s} \\ \lim_{c \rightarrow \infty} \left( \frac{dc}{d\phi} \right) &= \infty \end{aligned} \right\} \quad (4.16)$$

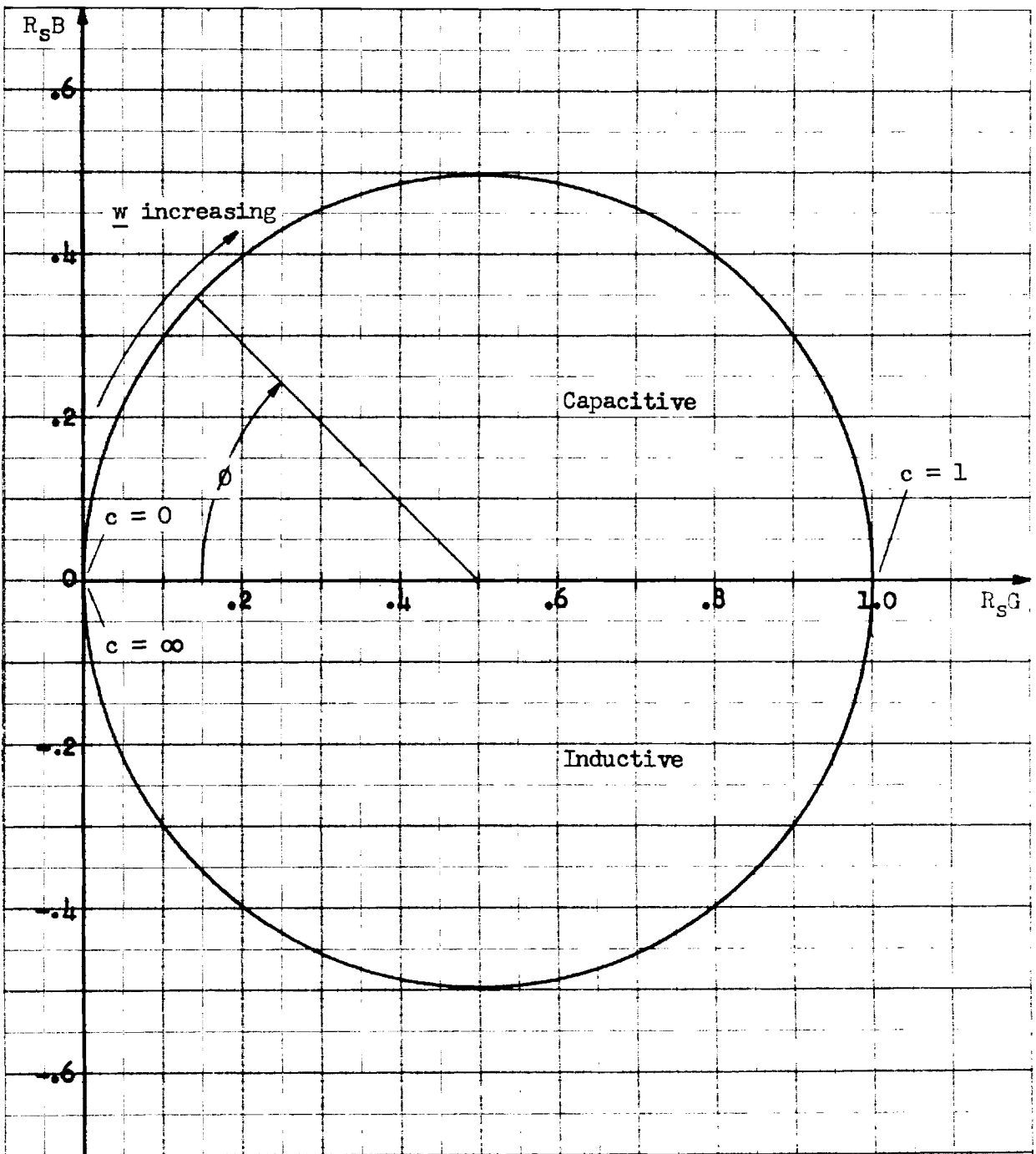


Figure 19. Polar Plot of Series RLC Admittance.

Figure 20 shows the variation of  $\underline{c}$  with  $\underline{\phi}$  for  $Q_s = 1, 10, \text{ and } 100$ . None of these values, of course, applies to a useful quartz resonator; however, it is easy to see from the curves that in the case of  $Q_s$  of the order of  $10^4$  or  $10^5$  the frequency scale will be extremely cramped near the origin and will change but slowly over the greater part of the circle in Figure 19.

The dotted vertical lines at 80 and 100 degrees on the lower scale of Figure 20 mark off a small region of frequencies in the neighborhood of maximum susceptance ( $\phi = 90^\circ$ ). The rate of change of frequency with respect to angle at maximum susceptance may be estimated by dividing the variation of  $\underline{c}$  in this interval by the 20-degree interval width.

It should be noted that the curves of Figure 20 describe  $\underline{c}$  as a function of  $0 \leq \phi \leq 180^\circ$  (left-hand and bottom scales), and give the reciprocal of frequency  $\frac{1}{c}$  as a function of  $180^\circ \leq \phi \leq 360^\circ$  (right-hand and top scales). In this way, the whole spectrum of positive real frequencies is covered by a single curve.

#### 4-4. Polar Curves of Total Admittance

Figure 21 shows the effect of the shunt capacitance on the series RLC admittance plot. The curves apply for  $Q_s = 1$ , and for  $N = 0.1$  and  $0.6$ . By comparison with the dotted curve, which applies for  $N = 0$ , it is seen that the addition of shunt capacitance has several marked effects. (1) Where for  $N = 0$  the polar plot was lineally finite, for  $N > 0$  it is neither finite in length nor bounded with regard to  $\underline{B}$ . (2) Where for  $N = 0$  there existed the single resonance  $c = 1$ , for  $N > 0$  two resonances occur—both at frequencies greater than  $c = 1$ . (3) As  $\underline{N}$  is increased the resonances approach one another, and coalesce when  $N = \frac{1}{3}$ , for the

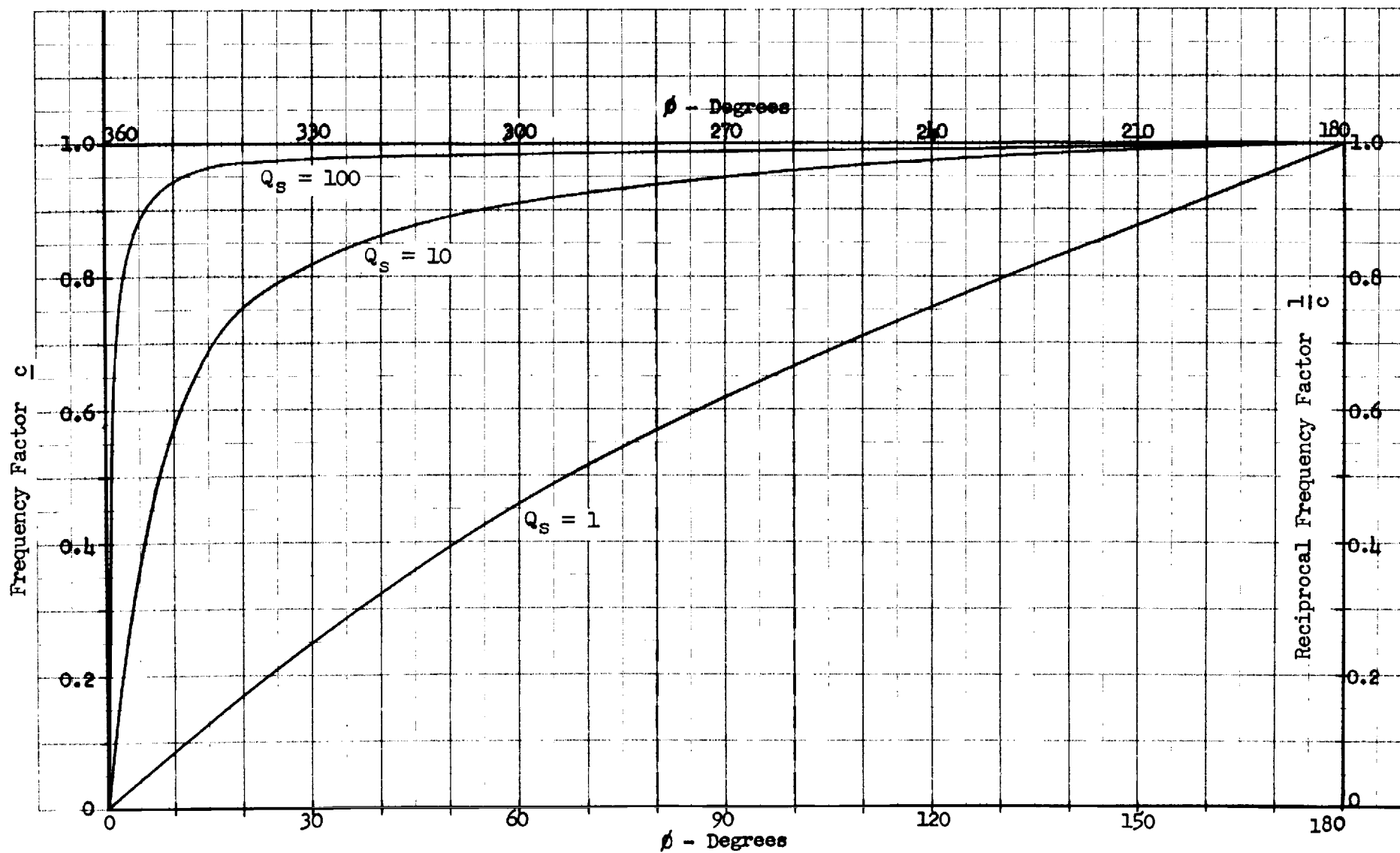


Figure 20. Frequency Scales for Polar Plot of RLC Admittance.

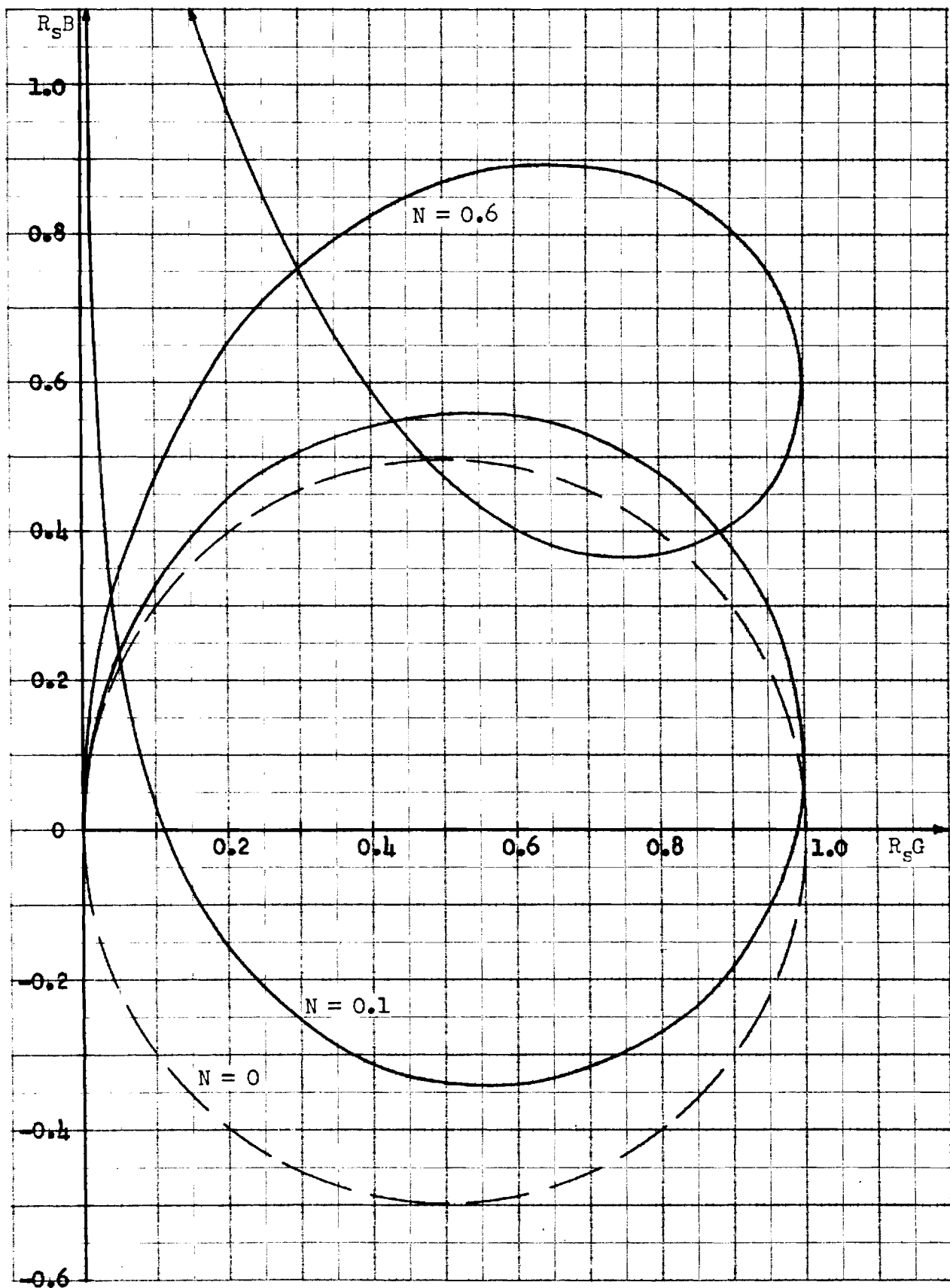


Figure 21. Polar Admittance Plot of RLC Branch with Shunt Capacitance.  $Q = 1$ .

case of  $Q_S = 1$ . (4) As  $N$  is further increased no resonances exist, and the crystal behaves more and more like a pure capacitance. The maximum of conductance, however, is always  $G = \frac{1}{R_S}$  and always occurs at the frequency  $c = 1$ .

Figure 22 shows the effect of the shunt capacitance when  $Q_S = 100$ . This value is still not typical of crystals, but illustrates the trend for increasing  $Q_S$ . Note that the inherent circular shape of the RLC admittance is retained as  $Q$  becomes larger. It is readily seen from Figure 20 that in the limit as  $Q_S$  approaches infinity, the frequency change from zero to one occurs within an infinitesimal increase of  $G$  at the origin. The frequency  $c$  then has unit value over the whole RLC admittance circle. The net effect of the shunt path, therefore, as  $Q_S$  increases without bound, is merely to shift the center of the circle to the point  $R_S G = 0.5$ ,  $R_S B = N$ . For finite  $Q_S$ , the low frequency end of the circle must, of course, remain connected to the origin (0,0); and the high frequency end must be connected to the "point" (0,  $\infty$ ).

For quartz crystals, where  $Q_S$  is typically  $10^5$ , the polar admittance curve is essentially the same as for infinite  $Q_S$ . The portion of this curve which is of interest in crystal analysis may be defined as the approximately circular part attached to the point where the curve crosses itself.

#### 4-5. Resonance Coalescence

It follows from the preceding remarks that, since the RLC admittance circle has a radius of 0.5, the coalescence of the resonances occurs for  $N = 0.5$  when  $Q_S$  is large. While it is not usual to operate crystals with shunt capacitances capable of giving a normalized capaci-

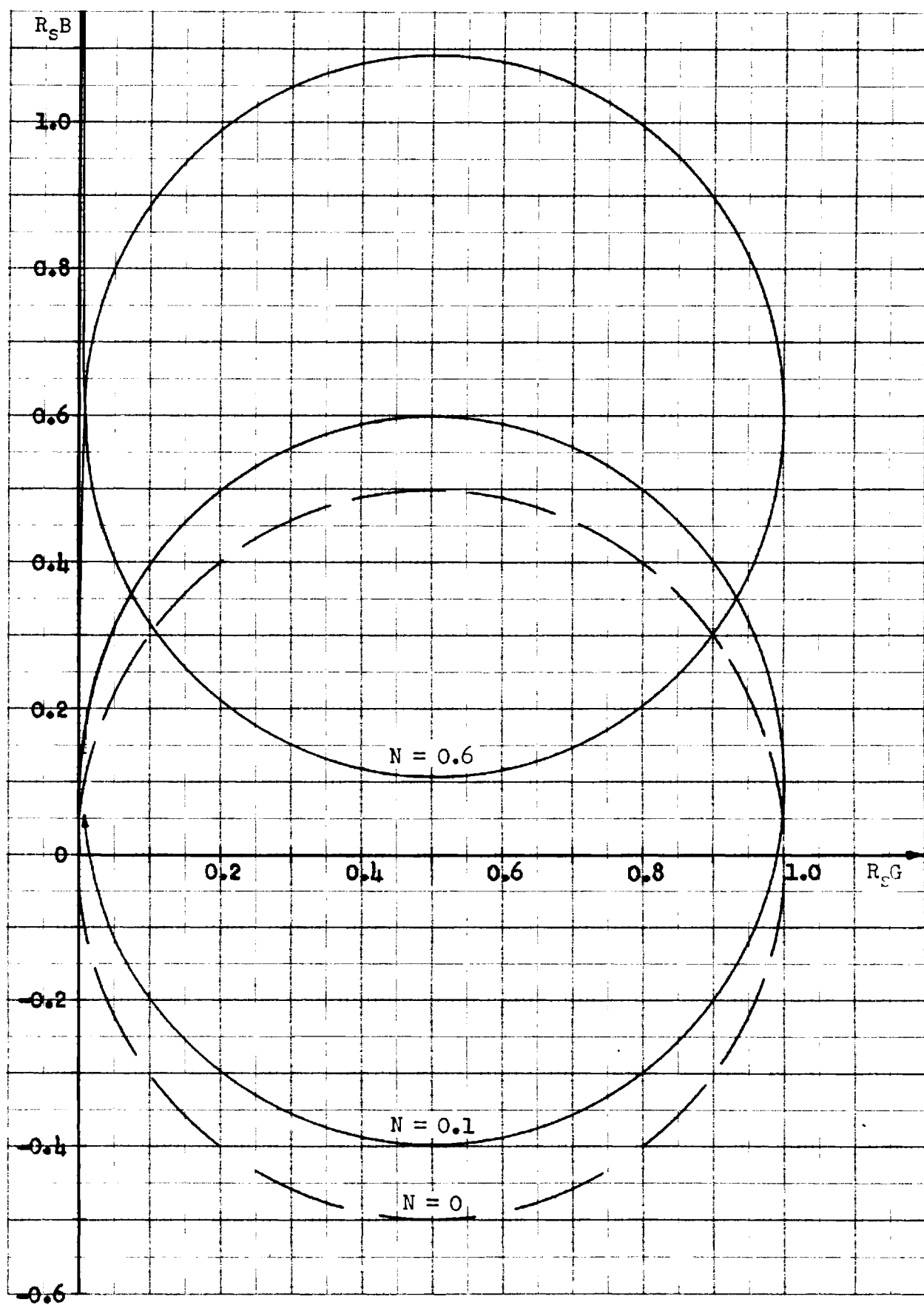


Figure 22. Polar Admittance Plot of RLC Branch with Shunt Capacitance.  $Q = 100$ .



tance ratio as large as 0.5, the merging of the series and parallel resonances of a crystal-like network is a matter of at least academic importance. It will be interesting to observe to what extent resonance coalescence depends on  $\underline{N}$  and  $\underline{Q}_s$ .

As shown in Appendix A, the condition for the existence of a resonance is:

$$\underline{N} \leq \frac{1}{2(1 + \frac{1}{2Q_s})} \quad (4.17)$$

The equality sign evidently defines the coalescence of the resonances. Figure 23 is, therefore, a plot of  $\underline{N}$  versus  $\underline{Q}_s$  for this condition. As mentioned in Section 4-3, for  $Q_s = 1$  the resonances merge when  $\underline{N} = \frac{1}{3}$ . It is seen that in the region of  $\underline{Q}_s$  for crystals ( $Q_s > 1000$ ),  $\underline{N}$  for coalescence is essentially 0.5. When  $\underline{N}$  is made greater than this value, the polar admittance plot cannot cross the conductance axis and the crystal can never become inductive. (Comparison should be made between this treatment and that given in Section 3-3.)

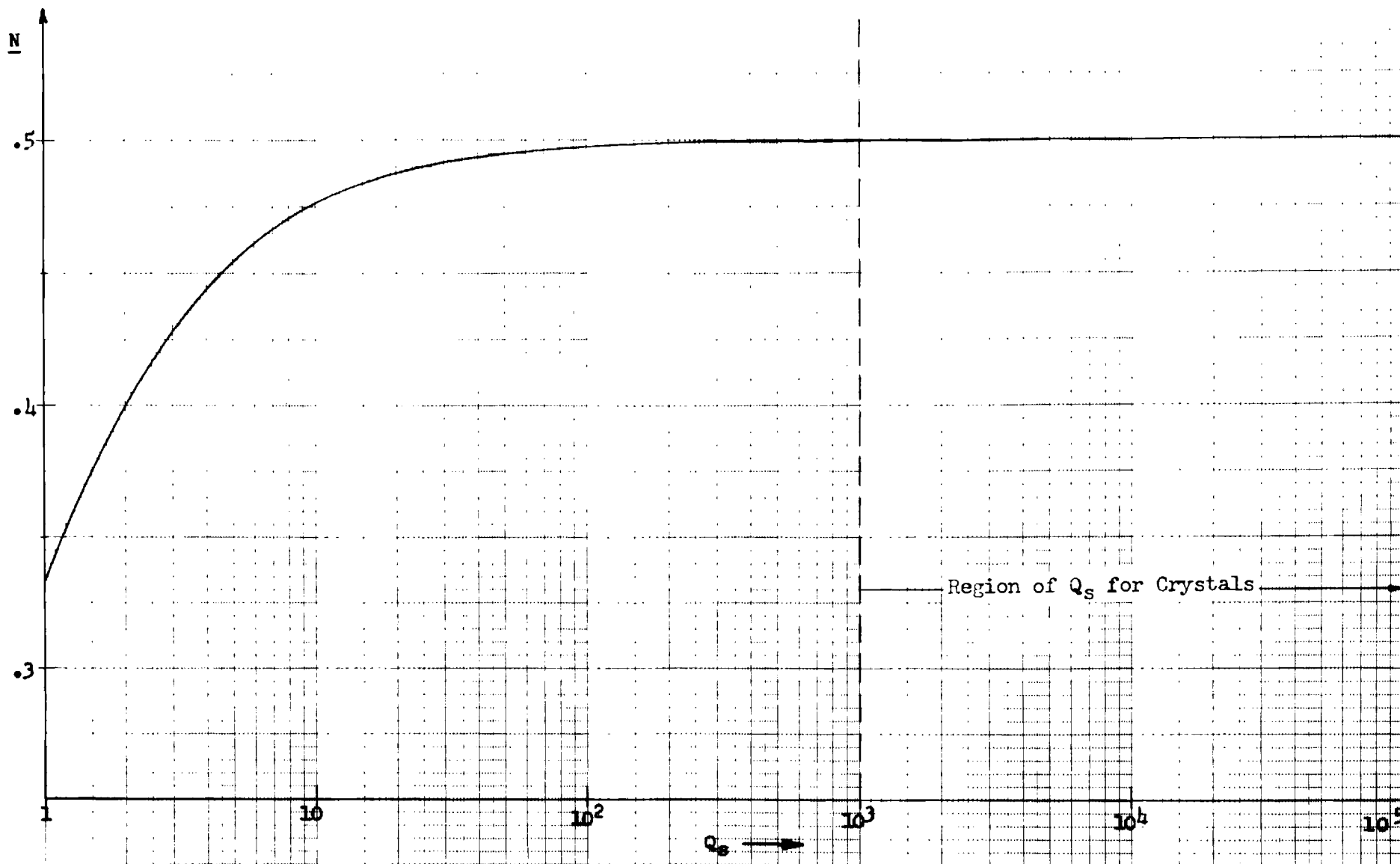


Figure 23. Normalized Capacitance Ratio Required to Produce Resonance Coalescence, as a Function of Crystal  $Q$ .

## V. DERIVATIVE FUNCTIONS OF A CRYSTAL

In Chapter III use was made of the slopes of the reactance function of a crystal at series and parallel resonant frequencies. The rate at which a function changes as one of its independent variables changes is, in general, valuable information. The present chapter is concerned with a study of several different derivative relationships which can be formed from the crystal impedance functions. Some of these find application in connection with problems on crystal-controlled oscillators. Others are included merely for the sake of completeness.

### 5-1. Derivatives with Respect to Frequency

Before proceeding with special derived impedance functions, it will be well to examine the formulation of derivatives with respect to frequency for the several frequency variables employed in this paper. Assuming that it is always possible, for our purposes, to regard a derivative as the ratio of one differential to another differential, we need only show the relationships between the differentials of the various frequency variables.

Having reference to the definitions of Chapter I, it is clear that:

$$dw = w_s dc = w_s d(1 + d) = w_s dd = \frac{w_s}{Q_s} dD \quad (5.1)^{13}$$

Often, derivatives with respect to the logarithm of frequency are more useful than those with respect to frequency itself. The equiva-

---

<sup>13</sup>Note that the differential operator and the fractional frequency deviation are represented by the same symbol  $d$ . The functional positions of the two quantities should obviate possible confusion between them.

lences here are:

$$d(\ln w) = \frac{dw}{w} = d(\ln cw_s) = d(\ln w_s) + d(\ln c) = d[\ln(1 + d)] \quad (5.2)$$

Now, the series expansion for the logarithm of unity plus a variable is:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (5.3)$$

When the variable is small, second and higher degree terms of the series may be neglected. As pointed out earlier,  $\underline{d}$  is restricted in crystal analysis to values small compared to unity. Hence, (5.2) becomes:

$$d(\ln w) \doteq d\bar{d} = \frac{1}{Q_s} d\bar{D} \quad (5.4)$$

Thus, from (5.4), derivatives taken with respect to  $\underline{D}$  and multiplied by  $\underline{Q}_s$  are equivalent to derivatives taken with respect to  $(\ln w)$ . Similarly, from (5.1), derivatives taken with respect to  $\underline{D}$  are equivalent to derivatives taken with respect to  $\underline{w}$  and multiplied by  $\frac{w_s}{Q_s}$ .

## 5-2. Derivatives with Respect to Phase Angle

Since the tangent of the impedance phase angle  $\underline{\theta}$  has been used in this paper as an independent variable, of comparable importance to frequency, derivatives with respect to the tangent,  $\underline{u}$ , are of interest.  $\underline{u}$  and  $\underline{\theta}$  are related by the equation:

$$u = \tan \theta \quad (5.5)$$

Taking the differentials of both sides,

$$du = d(\tan \theta) = \sec^2 \theta \cdot d\theta = (1 + \tan^2 \theta) \cdot d\theta = (1 + u^2) \cdot d\theta \quad (5.6)$$

Thus, derivatives with respect to  $\underline{\theta}$  are equivalent to derivatives taken with respect to  $\underline{u}$  and multiplied by the factor  $(1 + u^2)$ .

### 5-3. Derivatives with Respect to External Shunt Capacitance

Inasmuch as the characteristic behavior of a crystal may be altered by changes in the amount of capacitance  $\underline{C_x}$  shunting its terminals, derivatives with respect to  $\underline{C_x}$  prove useful. Noting that  $\underline{C_o}$  is a true constant, the following relationships are determined:

$$dC_x = d(C_y - C_o) = dC_y = C_s dn = C_s Q_s dN \quad (5.7)$$

Hence, derivatives with respect to  $\underline{C_x}$  are given through dividing derivatives taken with respect to  $\underline{N}$  by the factor  $C_s Q_s$ .

Similarly, since

$$d(\ln C_x) = \frac{dC_x}{C_x} = \frac{C_s Q_s}{C_x} dN \quad (5.8)$$

logarithmic derivatives are obtainable merely through further multiplication by  $\underline{C_x}$  of the derivatives taken with respect to  $\underline{N}$ .

Where the logarithmic derivatives with respect to the total shunt capacitance are desired, the results are even simpler:

$$d(\ln C_y) = \frac{dC_y}{C_y} = \frac{C_s Q_s}{C_y} dN = \frac{dN}{N} \quad (5.9)$$

### 5-4. Rate of Change of Phase Angle with Frequency

From (2.25),

$$\frac{du}{dD} = -8ND + 2 = 2(1 - 4ND) \quad (5.10)$$

This is the equation of a straight line having a slope of  $-8N$  and an

ordinate-axis intercept of + 2.

Substituting for  $\underline{D}$  from (2.24), the derivative is expressed as a function of  $\underline{u}$ :

$$\frac{du}{dD} = \mp 2\sqrt{1 - 4N(N + u)} \quad (5.11)$$

The derivatives with respect to other forms of the frequency variable are obtained through multiplication by the proper constant factor, as outlined in Section 5-1. The inverse derivative  $\frac{dD}{du}$  is in each case just the reciprocal of the expression given for  $\frac{du}{dD}$ .

Some of the family of curves represented by (5.10) are shown in Figure 24.

#### 5-5. Rate of Change of Frequency with Shunt Capacitance

From (2.24),

$$\frac{dD}{dN} = -\frac{1}{4N^2} \left[ 1 \pm \frac{1 - 2Nu}{\sqrt{1 - 4N(N + u)}} \right] \quad (5.12)$$

Substitution of (2.25) for  $\underline{u}$  in (5.11) gives the derivative in terms of frequency:

$$\frac{dD}{dN} = \frac{(1 - 2ND)^2 + N^2}{2N^2(4ND - 1)} \quad (5.13)$$

From (5.12), and employing the results of Sections 5-1 and 5-3, a very useful logarithmic formula is obtained:

$$\frac{d(\ln w)}{d(\ln C_y)} = \frac{N}{Q_s} \left( \frac{dD}{dN} \right) = \frac{(1 - 2ND)^2 + N^2}{2NQ_s(4ND - 1)} \quad (5.14)$$

Some of the family of curves defined by (5.14) are shown plotted in Figure 25.

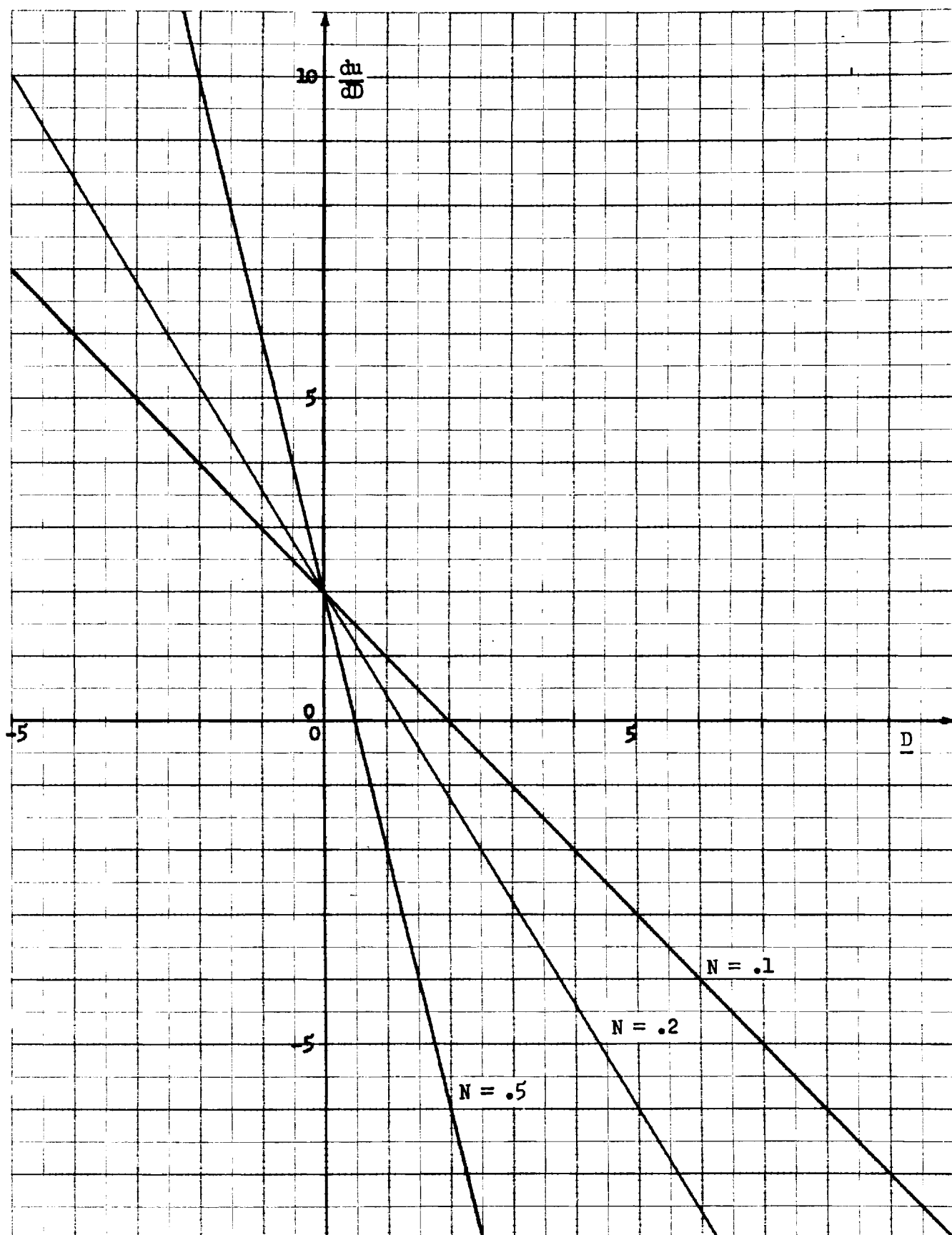


Figure 24. Rate of Change of Phase-Angle Tangent with Frequency, for Various Values of  $N$ .

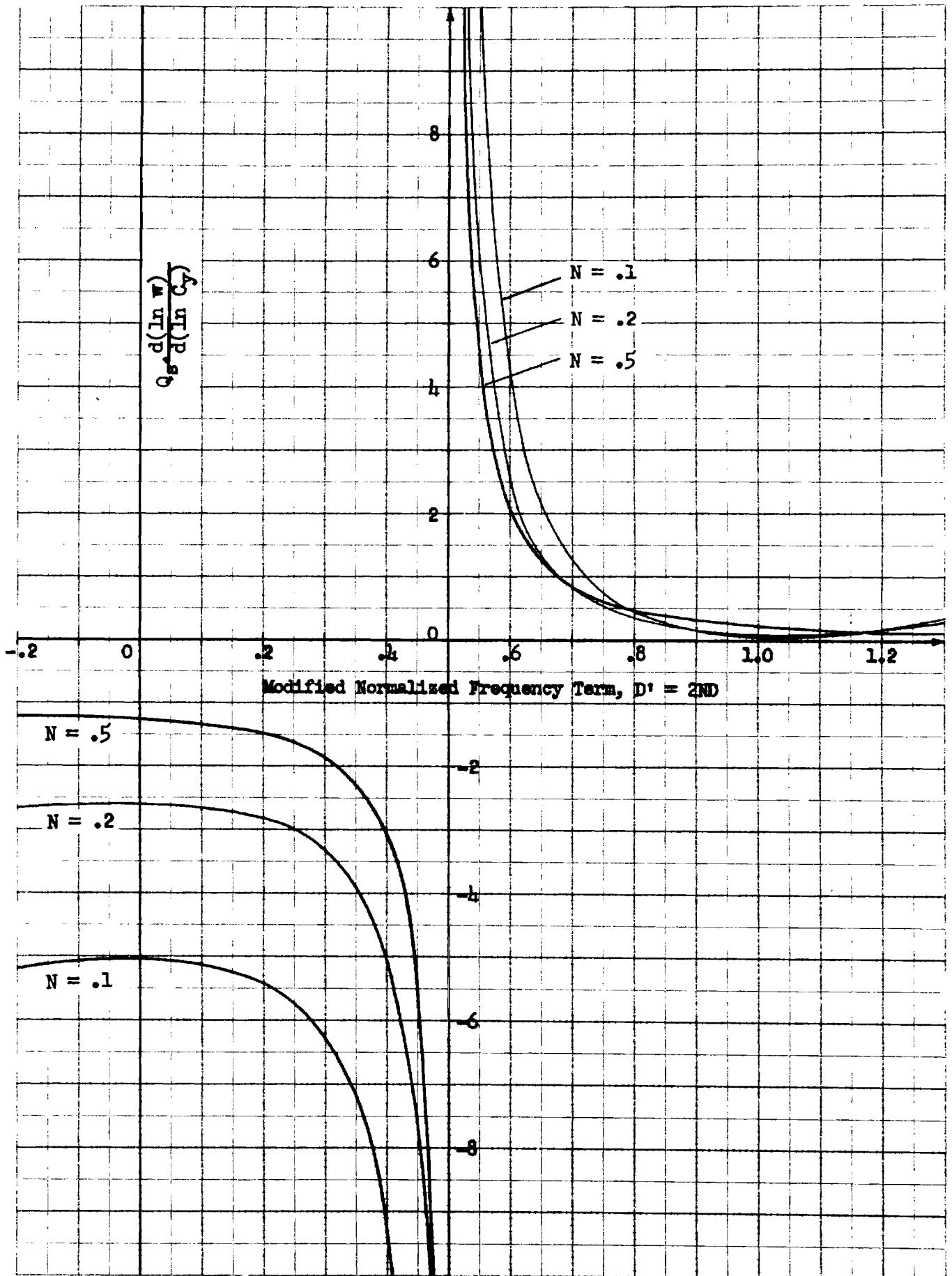


Figure 25. Rate of Change of Log-Frequency with Log-Shunt-Capacitance, for Various Values of  $N$ .



### 5-6. Rate of Change of Resistance with Frequency

From (2.31), denoting  $R(w)$  by  $\underline{R}$ :

$$\frac{dR}{dD} = \frac{4N(1 - 2ND) R_s}{[(1 - 2ND)^2 + N^2]^2} = \frac{4N(1 - 2ND) \cdot R^2}{R_s} \quad (5.15)$$

The derivative of the logarithm of resistance will also be found useful:

$$\begin{aligned} \frac{d(\ln R)}{dD} &= \frac{1}{R} \cdot \frac{dR}{dD} = 4N(1 - 2ND) \cdot \frac{R}{R_s} \\ &= \frac{4N(1 - 2ND)}{(1 - 2ND)^2 + N^2} \end{aligned} \quad (5.16)$$

### 5-7. Rate of Change of Reactance with Frequency

Here we solve at once for the logarithmic derivative. From (2.27), denoting  $X(w)$  by  $\underline{X}$ :

$$\frac{1}{\underline{X}} \frac{dX}{dD} = \frac{d(\ln X)}{dD} = \frac{d(\ln uR)}{dD} = \frac{d(\ln u)}{dD} + \frac{d(\ln R)}{dD} = \frac{1}{u} \frac{du}{dD} + \frac{1}{R} \frac{dR}{dD} \quad (5.17)$$

Using (2.25) and (5.10),

$$\frac{1}{u} \frac{du}{dD} = \frac{2(1 - 4ND)}{2D(1 - 2ND) - N} \quad (5.18)$$

Combining (5.18) and (5.16),

$$\frac{d(\ln X)}{dD} = \frac{2[(1 - 2ND)^2 - N^2]}{[2D(1 - 2ND) - N][(1 - 2ND)^2 + N^2]} \quad (5.19)$$

Multiplying through by

$$X = uR = \frac{[2D(1 - 2ND) - N] R_s}{(1 - 2ND)^2 + N^2} \quad (5.20)$$

gives the direct derivative:

$$\begin{aligned}
 \frac{dY}{dD} &= \frac{2 \left[ (1 - 2ND)^2 - N^2 \right] \cdot R_s}{\left[ (1 - 2ND)^2 + N^2 \right]^2} = 2 \left[ (1 - 2ND)^2 - N^2 \right] \cdot \frac{R_s^2}{R_s} \\
 &= \frac{(1 - 2ND)^2 - N^2}{2N(1 - 2ND)} \cdot \frac{dR}{dD}
 \end{aligned} \tag{5.21}$$

### 5-8. Rate of Change of Impedance Magnitude with Frequency

Again we set up the logarithmic derivative first. From (2.28),

$$\begin{aligned}
 \frac{1}{|Z|} \cdot \frac{d|Z|}{dD} &= \frac{d(\ln |Z|)}{dD} = \frac{d \ln (\sqrt{1 + u^2} \cdot R)}{dD} \\
 &= \frac{1}{\sqrt{1 + u^2}} \frac{d(\sqrt{1 + u^2})}{dD} + \frac{1}{R} \frac{dR}{dD}
 \end{aligned} \tag{5.22}$$

By the chain rule for derivatives, the first term in (5.22) may be written, using (5.11) in the last step:

$$\begin{aligned}
 \frac{1}{\sqrt{1 + u^2}} \frac{d(\sqrt{1 + u^2})}{dD} &= \frac{1}{\sqrt{1 + u^2}} \cdot \frac{d(\sqrt{1 + u^2})}{du} \cdot \frac{du}{dD} \\
 &= \frac{u}{1 + u^2} \cdot \frac{du}{dD} = \frac{\mp 2u\sqrt{1 - 4N(N + u)}}{1 + u^2}
 \end{aligned} \tag{5.23}$$

In order to combine this result with the second term in (5.22) the logarithmic resistance derivative (5.16) must be expressed in terms of  $u$ . This is done by means of (2.24).

$$\frac{1}{R} \frac{dR}{dD} = \frac{4N \left[ 1 \mp \sqrt{1 - 4N(N + u)} \right]}{1 - 2Nu \mp \sqrt{1 - 4N(N + u)}} \tag{5.24}$$

Substituting (5.23) and (5.24) into (5.22), and simplifying, there results finally:

$$\frac{d(\ln |Z|)}{dD} = \frac{2(u + 2N)}{1 + u^2} \tag{5.25}$$

The derivative may be expressed in terms of  $\underline{D}$  by substituting for  $\underline{u}$  from (2.25). However, the form is complicated, and it is probably preferable to retain  $\underline{u}$  as a parametric variable.

The direct derivative of impedance magnitude is found by multiplying through (5.25) by  $|Z|$  from (2.28):

$$\frac{d|Z|}{dD} = \frac{2(u + 2N)}{\sqrt{1 + u^2}} \cdot R \quad (5.26)$$

#### 5-9. Rate of Change of Phase Angle with Impedance Level.

As a final example of a derived crystal function, we shall consider the derivative

$$|Z| \cdot \frac{d\theta}{d|Z|} = \frac{d\theta}{d(\ln |Z|)} = \frac{d\theta}{du} \cdot \frac{du}{dD} \cdot \frac{dD}{d(\ln |Z|)} \quad (5.27)$$

The practical importance of logarithmic derivatives lies in the fact that they express percentile or fractional rates of change. Equation (5.14), for instance, may be interpreted as a statement of the per cent change in frequency resulting from a per cent change in shunt capacitance. The derivative now under consideration expresses the variation in crystal phase angle which occurs when the impedance level is changed by a certain fraction  $\frac{d|Z|}{|Z|}$ .

Equation (5.27) may be expanded by means of (5.6), (5.10), and (5.25) to yield:

$$\frac{d\theta}{d(\ln |Z|)} = \frac{1 - 4ND}{u + 2N} \quad (5.28)$$

Upon substitution for  $\underline{u}$  from (2.25), there is obtained a form containing

the single independent variable  $\underline{D}$  and the parameter  $\underline{N}$ .

$$\frac{d\theta}{d(\ln |Z|)} = \frac{1 - 4ND}{2D(1 - 2ND) + N} \quad (5.29)$$

## VI. APPLICATIONS

The use of piezoelectric crystals in oscillator circuits has, in the past, been restricted largely to cases where the shunting capacitance is small. First approximations to such critical quantities as the equivalent crystal resistance at antiresonance have been well understood and widely used in specifying the properties of crystals for particular applications. The formulas presented in the preceding chapters may be regarded as second approximations to the true impedance characteristics of crystals. It has been pointed out from time to time in the development how these more exact expressions may legitimately be reduced to conventional forms whenever the practical capacitance ratio of the crystal is in value some small fraction of the crystal  $Q$ . However, the effect of increasing this normalized capacitance ratio beyond 0.1 is very marked and requires modification of the usual indices of crystal performance.

The mathematical theory of the present paper is capable of handling the less restricted situations without further extension. The only limitations which have been applied are: (1), that the  $Q$  of the series arm of the crystal be large compared to unity, a condition invariably satisfied with quartz crystals (indeed, it is chiefly for the sake of their high  $Q$ 's that crystals are used as circuit elements); and (2), that the minimum intrinsic capacitance ratio of crystals be of the order of 100, a condition also satisfied by quartz.

### 6-1. General Effects of Shunt Capacitance on Crystal Performance

Because of the unavoidable holder capacitance, a crystal behaves

predominantly like a capacitive device. Within a narrow band of frequencies near each resonance there occurs a rise and fall of inductive properties, but over the infinite remainder of real frequencies the impedance phase angle is negative. By augmenting the amount of shunt capacitance present, one reduces the maximum positive phase angle at which the crystal may operate, and one decreases the frequency band over which inductive behavior occurs. However, it is an important fact that the intrinsic  $Q$  of the crystal is not materially degraded by this process. (See Appendix B.)

The use of crystals shunted by large values of padding capacitance constitutes a new and comparatively unexplored field of application. It is believed that the ability to control the impedance level of a crystal, without sacrifice of inherent high selectivity or  $Q$ , makes the variation of shunt capacitance an important subject for study.

## 6-2. Solution of Circuit Problems

The crystal formulas presented earlier are of immediate value in solving circuit problems involving crystals as component elements. Such quantities as the rate of change of phase angle with frequency, and the rate of change of frequency with impedance level, are obviously applicable in ascertaining the frequency stability of crystal-controlled oscillators.

In determining the conditions for oscillation of a crystal-controlled oscillator, one may add the impedance (given in Chapter II) of the crystal and associated shunt capacitance to the impedance of the remainder of the circuit. Equating the sum to zero yields two simultaneous conditional equations, which may be solved for the frequency of operation, the minimum transconductance or amplification factor required for oscil-

lation—or any other pair of unknowns.

### 6-3. Methodology

Perhaps the most important aspect of this paper is that it provides a basis from which to continue mathematical investigations of crystal performance and application. The derived functions of Chapter V illustrate rather than embrace the crystal derivatives of interest. However, the basic formulas are recorded in the earlier chapters, and from them all else is deducible.

## BIBLIOGRAPHY

Cady, W. G., Piezoelectricity, New York: McGraw-Hill Book Company, 1946.

Edson, W. A., et al., The Keying Properties of Quartz Crystal Oscillators, Final Report, Project No. 106-6, State Engineering Experiment Station, Georgia Institute of Technology, Atlanta, December, 1947.

Heising, R. A., Quartz Crystals for Electrical Circuits, New York: D. Van Nostrand Company, 1946.



## APPENDIX A

## CRYSTAL RESONANCES

The terms resonance (or series resonance) and antiresonance (or parallel resonance), as applied to an electrical circuit, refer to conditions such that the equivalent series impedance of the circuit becomes a pure real number—or, what is equivalent, conditions which produce zero impedance phase angle. The resonant frequencies  $\underline{w}_r$  and  $\underline{w}_a$  of a crystal may, thus, be determined by setting  $\underline{u}$  equal to zero in Equation (2.7), which relates frequency with phase angle. There results a quadratic in the square of the frequency factor  $c = \frac{w}{w_s}$ :

$$c^4 - c^2(m^2 + 1 - \frac{1}{Q_s^2}) + m^2 = 0 \quad (\text{A.1})$$

The solutions are:

$$c^2 = \frac{1}{2}(m^2 + 1 - \frac{1}{Q_s^2}) \pm \frac{1}{2} \sqrt{\Delta(c)} \quad (\text{A.2})$$

where

$$\Delta(c) = (m^2 + 1 - \frac{1}{Q_s^2})^2 - 4m^2 \quad (\text{A.3})$$

Inasmuch as for real frequencies the discriminant (A.3) must be positive or zero, the following inequality holds:

$$(m^2 - 1)^2 - \frac{2(m^2 + 1)}{Q_s^2} + \frac{1}{Q_s^4} = \Delta(c) \geq 0 \quad (\text{A.4})$$

Since  $m^2 = 1 + \frac{1}{n}$  and  $n = NQ_s$ , (A.4) can be rearranged to a quadratic inequality in  $\underline{N}$ :

$$4N^2(1 - \frac{1}{4Q_s^2}) + \frac{2N}{Q_s} - 1 \leq 0 \quad (\text{A.5})$$

From (A.5) we obtain the requirement for resonance,

$$N \leq \frac{1}{2(1 + \frac{1}{2Q_s})} \quad (\text{A.6})$$

In reducing the solutions (A.2) to a practical form, it might appear that terms in  $\frac{1}{Q_s^2}$  could immediately be neglected, because of the invariably high values of  $Q$  which exist for quartz crystals. But if such an approximation is made, only the first term in (A.4) will be retained. Since this first term is equal to  $\frac{1}{n^2}$ , the effect of discarding the second term will be considerable unless  $n$  is very much smaller than  $Q_s$ —a restriction which it is not desirable to apply at the present stage.

The final term in (A.4) may, however, be dropped with insignificant error. The result is:

$$\Delta(c) = \frac{1}{n^2} \left[ 1 - \frac{2n}{Q_s^2} (2n - 1) \right] \quad (\text{A.7})$$

Since  $n$  is invariably greater than 100 for quartz crystals, we may neglect unity compared with  $2n$  in the final term of (A.7). Omitting also the term  $\frac{1}{2Q_s^2}$  in (A.2), and introducing  $N = \frac{n}{Q_s}$ , we obtain:

$$c^2 \doteq 1 + \frac{1}{2n} (1 \pm \sqrt{1 - 4N^2}) \quad (\text{A.8})$$

Again, since  $n$  is large, the square root of (A.8) may be taken to two terms of the Binomial Theorem expansion. Thus

$$c \doteq 1 + \frac{1}{4n} (1 \pm \sqrt{1 - 4N^2}) \quad (\text{A.9})$$

Where  $N$  is very small compared to one, the binomial approximation

is again applied to yield:

$$c \doteq 1 + \frac{1}{4n} \left[ 1 \pm (1 - 2N^2) \right] \quad (\text{A.10})$$

We then have, for  $\underline{N}$  small:

$$c_a \doteq 1 + \frac{(1 - N^2)}{2n} \doteq 1 + \frac{1}{2n} \quad (\text{A.11})$$

$$c_r \doteq 1 + \frac{N^2}{2n} \doteq 1 \quad (\text{A.12})$$

Noting that

$$m = \sqrt{1 + \frac{1}{n}} \doteq 1 + \frac{1}{2n} \quad (\text{A.13})$$

we write

$$w_a = c_a w_s \doteq m w_s \quad (\text{A.14})$$

$$w_r = c_r w_s \doteq w_s \quad (\text{A.15})$$

## APPENDIX B

## THE "Q" OF A CRYSTAL

One of the more general definitions for the quality factor or selectivity,  $\underline{Q}$ , of a circuit is:<sup>14</sup>

$$\underline{Q} = \frac{2\pi(\text{Energy Stored in the Circuit})}{(\text{Energy Lost per Cycle})} \quad (\text{B.1})$$

It has been shown by Edson<sup>15</sup> that, for a network which is characterized by a second order differential equation, the quality factor (B.1) becomes

$$\underline{Q} = \frac{\underline{B}}{2\underline{A}} \quad (\text{B.2})$$

in which  $\underline{A}$  and  $\underline{B}$  are the real and imaginary components, respectively, of the complex conjugate roots of the "associated algebraic equation." ( $\underline{A}$  is called the "time decrement," and  $\underline{B}$  is the natural resonant frequency of the network.)

Edson further points out that the above definition is meaningless when applied to a combination of elements which is incapable of natural oscillation. However, it is common engineering practice to ascribe quality factors to single physical circuit elements, since an indication is thereby provided of the relative dissipation which may be expected in

---

<sup>14</sup>F. E. Terman, Radio Engineers' Handbook (New York: McGraw-Hill Book Company, 1943), p. 269.

<sup>15</sup>W. A. Edson, et al., The Keying Properties of Quartz Crystal Oscillators (Final Report, Project No. 106-6, State Engineering Experiment Station, Georgia Institute of Technology, Atlanta, December, 1947), p. 28.

such elements when operated at a particular frequency. The definition of the engineering  $Q$  of an inductance, or of a combination of elements containing inductance, has the general form:

$$Q' = \frac{\omega L}{R} \quad (B.3)$$

It will be interesting to compare the results of applying the above two definitions, (B.1) and (B.3), to crystals.

In the notation of algebraized Heaviside calculus, a crystal consists of two impedance branches as follows:

$$z_s(p) = pL_s + R_s + \frac{1}{pC_s} \quad (B.4)$$

$$z_y(p) = \frac{1}{pC_y} \quad (B.5)$$

(Note that these two expressions are obtainable merely by substituting  $p = j\omega$  in the ordinary steady-state complex-algebra impedance formulas for the two branches.) The total  $p$ -impedance around the closed path containing all the elements is the sum,

$$\begin{aligned} z_1 = z_s + z_y &= \frac{L_s}{p} \left( p^2 + \frac{R_s p}{L_s} + \frac{C_y + C_s}{L_s C_s C_y} \right) \\ &= \frac{L_s}{p} (p^2 + 2A_s p + m^2 \omega_s^2) \end{aligned} \quad (B.6)$$

where

$$\left. \begin{aligned} A_s &= \frac{R_s}{2L_s} \\ \omega_s^2 &= \frac{1}{L_s C_s} \\ m^2 &= 1 + \frac{C_s}{C_y} = 1 + \frac{1}{n} \end{aligned} \right\} \quad (B.7)$$

If a spectrum of noise frequencies is impressed on the crystal—say, by giving it a sudden blow—oscillations will persist longest at that frequency for which the impedance around the closed path is least. Inasmuch as negative resistance is not possible with purely passive elements, the minimum impedance will be  $z = 0$ .

The result of substituting this condition in (B.6) is the "associated algebraic equation" of the network. The desired roots are easily found to be:

$$p = A_1 \pm jB_1 = -A_s \pm j\sqrt{m^2 w_s^2 - A_s^2} \quad (\text{B.8})$$

Hence, from (B.2),

$$Q_1 = \frac{\sqrt{m^2 w_s^2 - A_s^2}}{2A_s} \quad (\text{B.9})$$

Using the definition of (1.3) in conjunction with those of (B.5), we may write:

$$\frac{w_s}{A_s} = 2Q_s \quad (\text{B.10})$$

Then (B.9) becomes:

$$Q_1 = mQ_s \sqrt{1 - \frac{1}{4m^2 Q_s^2}} \quad (\text{B.11a})$$

Since  $Q_s$  for crystals is extremely large, and since  $m$  is a number greater than one, (B.11a) can be reduced to:

$$Q_1 = mQ_s = \frac{mw_s L_s}{R_s} = \frac{w_a L_s}{R_s} \quad (\text{B.11b})$$

the last step being in accordance with Appendix A.

If the conditions of antiresonance are employed in (B.3), we obtain the expression

$$Q_1' = \frac{w_a L_s}{R(w_a)} \quad (\text{B.12})$$

Using the approximate formula for  $R(w_a)$  when  $\underline{N}$  is small, as given by (3.7b), (B.12) becomes:

$$Q_1' = \frac{m w_s L_s N^2}{R_s} = N^2 (m Q_s) \quad (\text{B.13})$$

This engineering  $\underline{Q}$  at antiresonance is seen to be smaller than the natural  $\underline{Q}$  given by (B.11b), in the ratio of  $N^2:1$ .

Next, let us short-circuit the crystal terminals, thereby obtaining in place of (B.6) the equation

$$z_2 = z_s = \frac{L_s}{p} (p^2 + 2A_s p + w_s^2) \quad (\text{B.14})$$

The roots of the associated algebraic equation of the circuit in this case are

$$p = A_2 \pm jB_2 = -A_s \pm j\sqrt{w_s^2 - A_s^2} \quad (\text{B.15})$$

Thus,

$$Q_2 = \frac{B_2}{A_2} = Q_s \sqrt{1 - \frac{1}{4Q_s^2}} \doteq Q_s \quad (\text{B.16})$$

The engineering  $\underline{Q}$  at resonance is

$$Q_2' = \frac{w_r L_s}{R(w_r)} \quad (\text{B.17})$$

Using (3.6b) for  $\underline{R}$  in conjunction with the approximation for  $\underline{w_r}$  derived in Appendix A, there results:

$$Q_2' \doteq \frac{\omega_s L_s}{R_s} = Q_s \quad (\text{B.18})$$

It is well to point out that the  $\underline{Q}$  defined by (B.12) is physically not too significant and is seldom used.<sup>16</sup> Moreover, that given by (B.17) is more usually replaced by the approximate  $\underline{Q}$  of the series RLC branch of the crystal. Thus, the quantity  $\underline{Q}_s$  which has been used throughout this paper may be considered the common engineering  $\underline{Q}$  of a crystal. The definition (B.1) is in many ways more rigorous and satisfactory than the rather arbitrary engineering definitions cited. However, it is seen from the derived expressions (B.11b) and (B.16) that, at least in the case of high- $\underline{Q}$  devices, the "energy-stored-and-lost" equation leads to forms which are substantially equivalent to the common  $\underline{Q}_s$ .

---

<sup>16</sup>W. G. Cady, Piezoelectricity (New York: McGraw-Hill Book Company, 1946), p. 496.